

  
**ANALYSIS OF A BOUNDARY VALUE PROBLEM FOR A SYSTEM OF  
NON-HOMOGENEOUS ORDINARY DIFFERENTIAL EQUATIONS  
(ODE), WITH VARIABLE COEFFICIENTS.**

By

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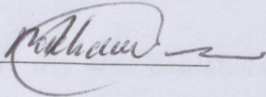


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## Declaration

I declare that this dissertation for the degree of Masters of Science in Mathematics (MSCMAD) in the Department of Mathematics and applied Mathematics, Faculty of Mathematical and Natural Sciences, University of Venda hereby Submitted has not been previously submitted elsewhere for a degree at this or any other University, and it is my own work in design and in execution, and that all materials consulted have been properly acknowledged.

Signature



24

day of October 2013

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## Dedications

The journey of a thousand of miles begins with a single step. This dissertation is dedicated to the following people:

- (i) My parents: Thank you for your unconditional support during my studies. I am honoured to have you as my parents. Thank you for giving me a chance to prove and improve myself through all my walks of life. Please do not ever change. I love you all.
- (ii) My family: My wife Sehlabela Portia who prayed for my success in education. She believes that “tough times do not last but tough people do”. My further dedications to my sons Cauchy Lwazi and Rayleigh Siyabonga. Thank you for believing in me and allowing me to further my studies. Please do not ever doubt my dedication and I love you all.
- (iii) Siblings: Hoping that with this research I have proven to you that there is no higher mountain to climb as long as God is on our side. Hoping that you will walk again and be able to fulfil your dreams.

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# Abstract Contents

In this study we present a condition for the existence and uniqueness of the solution  $y(x)$  for a system of nonhomogeneous linear first order Ordinary Differential Equations (ODE). The existence and uniqueness of the solution of  $y(x)$  was confirmed through the Picard Lindelof Theorem. We then study the stability of matrix  $A(x)$  using its spectrum, moreover,  $A(x)$  is symmetric. This is a pre-condition for the application of Lefschetz direct stability method. We then modify the given Lefschetz system (Meyer, 1964) to suit the problem at hand. The direct method requires the construction of a suitable Lyapunov function; not easy for a time-independent (non-dynamic) problem. For a time-dependent problem the energy thereof becomes a suitable candidate for a Lyapunov function. For a non-dynamic problem it is harder to construct a Lyapunov function as there are no rules for that purpose. In our study we modified the Lefschetz system for the direct stability method and applied it to confirm the Lefschetz stability criterion using the modified systems of linear first order ODEs with variable coefficients. The Lefschetz method afforded us the construction of a credible Lyapunov function which enabled us to confirm the stability of the null solution to our problem. From our modified Lefschetz direct stability system, we solved the Makhabane / Hlomuka equation (5) for  $B(x)$  (7) which we later confirmed as both symmetric and positive definite.

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# CHAPTER 1

## 1.1. Background to the study:

The history of ODEs goes all the way back to the seventeenth century when two great scientists Isaac Newton and Gottfried Leibniz introduced Calculus which had been developed from the concept of function (Kline, 1972). Differential equations are essential tools in physical problems adhere relevance in almost every sphere of human endeavor from Agricultural Sciences, Engineering, Medical Science, Physical Sciences, to Social Sciences. Among the earlier work on differential equations, the work of Euler and Lagrange theory of linear system of ODEs (Ogundare, 2009). In general, the study of non-linear first order ODE is restricted to a variety of special cases and the method of solution usually involves one or more of a limited number of different methods for the linear first order ODEs; it is possible to derive closed form expressions whereas this is not possible in general for the non-linear first order ODEs. A differential equation is an equation containing an unknown function and its derivatives as well as independent variables. To solve a differential equation is to find a function that, when substituted into the differential equation, produces an identity. Such a function is a solution to the differential equation. If the unknown function is a function of a single independent variable, then the differential equation is an ODE. If the unknown function is a function of more than one independent variable and partial derivatives occur in the equation, then it is a partial differential equation (PDE). In our studies, we will take a look at the first order ODEs of order  $n$  (Boyce & DiPloma, 1965) and (Gershenfeld, 1999). There are several theorems that establish existence and uniqueness of solutions to initial value problems involving ODEs both locally and globally. It would be an oversight to ignore the conditions which establish such existence and uniqueness. The Picard-Lindelof theorem shows how existence and uniqueness of solutions to certain initial value problems may be confirmed.

**Definition:** An initial value problem (also called a Cauchy problem) is a differential equation  $y'(t) = f(t, y(t))$  with  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  together with a point in the domain of  $g(t_0, y_0) \in \mathbb{R} \times \mathbb{R}$ , called the initial condition. Consider the initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0, \quad \text{for } t \in [t_0 - \varepsilon, t_0 + \varepsilon].$$

Suppose  $f$  is Lipschitz continuous in  $y$  and continuous in  $t$ . Then, for some value  $\varepsilon > 0$ , there exists a unique solution  $y(t)$  to the initial value problem with  $t$  in the range  $(t_0 - \varepsilon, t_0 + \varepsilon)$  (Coddington & Levinson, 1955).

A solution to an initial value problem is a function  $y(t)$  that can take values on infinite dimensional spaces, such as Banach spaces or spaces of distributions. A well understood particular class of differential equations is the class of linear differential equations. We can always reduce a system of differential equation to explicit vectorial linear differential equation of order 1 i.e.

$$y'_i(x) = \sum_{j=1}^n a_{ij}(x)y_j(x) + b_i(x) \quad i = 1, 2, \dots, n$$

$$\text{subject to} \quad : \quad y_j(0) = c_j, \quad j = 1, 2, \dots, n.$$

We can represent a system of differential equations of order 1 to a given interval when boundary conditions as a vectorial linear differential equation of order 1 with boundary condition. This can be write concisely using matrix and vector notation as

$$\frac{dY(x)}{dx} = A(x)y(x) + \mathbf{b}(x)$$

where

$$Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{pmatrix}, \quad A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{pmatrix}, \quad b(x) = \begin{pmatrix} b_1(x) \\ b_2(x) \\ \vdots \\ b_n(x) \end{pmatrix} \text{ and}$$

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix},$$

in an open bounded interval  $\Omega \subset \mathbb{R}^1$ , where  $b(x)$  is a variable and  $c$  is a constant vector. Having studied existence and uniqueness for problem (1), we will then turn to the stability of the null solution thereof. The study of stability will involve the construction of an appropriate Lyapunov function; not easy for a non-dynamic problem. However, there are other ways of constructing a Lyapunov function through the application of the Lefschetz direct method (Meyer, 1964).

**Definition:** A motion  $M$  is said to be Lyapunov stable if given any  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that if  $N$  is any motion which starts at  $t = 0$  inside a  $\delta$  - ball centered at  $M$  for all time  $t$ . In particular this means that an equilibrium point  $P$  will be Lyapunov stable if you can choose the initial conditions sufficiently close to  $P$  (inside a  $\delta$  - ball) so as to be able to keep all the ensuing motions inside an arbitrary small neighborhood of  $P$  (inside an  $\varepsilon$  - ball). A motion is said to be Lyapunov unstable if it is not Lyapunov stable.

## 1.2. Existence and uniqueness of the solution to the problem:

### 1.2.1. Introduction:

In this introductory chapter we define the class of ordinary differential equations, which will be considered in this dissertation and explain how they are used. In addition discuss how to find solutions of some differential equations of the first order. A differential equation is an equation relating a function and its derivatives. Our objective is to find a function satisfying the given equation. Let us recall that a first order initial value problem consists of a differential equation and specified initial conditions given at some value  $x_0$  of the independent variable; that is

$$\frac{dy}{dx} = f(x, y), \quad y(0) = \mathbf{c} \quad (1)$$

For several types of first order ODEs we can find general solutions and hence solve the initial value problem. However, there are many first order differential equations for which no solution in closed form can be found. In such cases the first thing we want to know is whether there is only one solution for a given initial condition. For this reason The Picard-Lindelof Theorem plays a central role. It tells us that solutions to first-order initial value problem exists and are unique under certain reasonable conditions. We have a unique solution of the differential equation in (1) if we specify the constant  $c$  in the initial data. One way to require that the function  $y(x)$  not only to satisfy the differential equation but also to satisfy the condition that it takes on a prescribed value  $y_0$  at a prescribed point  $x_0$ . We now consider the problem statement to a system of nonhomogeneous linear first order ODEs of the form:

$$\begin{aligned} \frac{dY(x)}{dx} &= A(x)y(x) + \mathbf{b}(x) \\ \text{subject to} & : Y(0) = \mathbf{c}, \end{aligned} \quad (2)$$

in an open bounded domain  $\Omega \subset \mathbb{R}^1$ ; with  $c$  constant  $1 \times n$  vector and where  $A(x)$  is a (potentially  $x$ -dependent) matrix and  $b(x)$  is some prescribed vector function of  $x$ .

Our study investigates the existence and uniqueness of the solutions  $y(x)$  to the problem. We only consider the case when the matrix  $A(x)$  is independent of time. A number of approaches are attempted. We also investigate necessary and sufficient conditions for the stability of the solution to the problem (2) by using Lyapunov theory. Application of this to the asymptotic stability of equilibria for vector fields is considered. We consider stability of matrices is intimately related to the stability of stationary point solutions in the theory and applications of dynamical systems. We also look into the construction of analytical solutions to the problem (2) into the simplest class of linear first order differential equations. Solutions of ordinary differential equations often involve a wide range of time scales. In many problems, one is only interested in the slow dynamics, or in the long-time behaviour of the solutions. In this study we deal with the nonhomogeneous system for a first order linear differential equations with variable coefficients. We establish sufficient conditions for all solutions to nonhomogeneous system of differential equation to be considered. We also give conditions under which there exists at least one nonhomogeneous system of the solution to our problem (2).

(c) There exists a unique and single-valued map,  $A: \Omega \rightarrow \mathbb{R}^n$  defined by

$$x \mapsto A(x), \text{ where } i = 1, 2, 3, \dots, n.$$

(d) For problem (1), it therefore makes sense to choose  $x \in S := \text{Dom}(g) \cap \text{Dom}(A) \subset \Omega$ .

(e) All the norms in this paper are  $\mathbb{R}^n$  and  $M_n^{\mathbb{R}^n}$ -norms, where  $M_n^{\mathbb{R}^n}$  is assumed to be a vector space for some  $x \in S$ .

## 1.2.2. Statement of the problem:

We consider the following problem:

$$\left. \begin{aligned} \frac{dY(x)}{dx} &= A(x)Y(x) + b(x) \\ \text{Subject to : } Y(0) &= \mathbf{c} \end{aligned} \right\} \text{linear system} \quad (3)$$

in an open bounded domain  $\Omega \subset \mathbb{R}^1$ ,  $x \in \Omega$  with  $\mathbf{c}$  as a constant  $1 \times n$  vector. In other words, for each  $x \in \Omega$ ,  $A(x)$  is an  $n \times n$  matrix.

## 1.2.3. Basic assumptions and preliminary notions:

In the analysis of the problem, we assume that the following holds:

### 1.2.3.1 Basic assumptions:

- (a) The matrix  $A(x) = (a_{ij}(x))$  is symmetric for all  $x \in \Omega$ ;
- (b) There exists a unique, multi-valued, bounded map,  $g : \Omega \rightarrow \mathbb{R}^1$  defined by  $x \rightarrow a_{ij}(x)$ , where,  $i, j = 1, 2, 3, \dots, n$ .
- (c) There exists a unique and single-valued map,  $h : \Omega \rightarrow \mathbb{R}^1$  defined by  $x \rightarrow b_i(x)$ , where,  $i = 1, 2, 3, \dots, n$ .
- (d) For problem (1), it therefore makes sense to choose  $x \in S := \text{Dom}(g) \cap \text{Dom}(h) \subset \Omega$ .
- (e) All the norms in this paper are  $\mathbb{R}^n$  and  $\mathbb{M}_{\mathbb{R}}^{n \times n}$  - norms, where  $\mathbb{M}_{\mathbb{R}}^{n \times n}$  is assumed to be a vector space for some  $x \in S$ .

### 1.2.3.2. Preliminary notions:

#### 1.2.3.2.1. The norm of the matrix $A(x)$ :

An  $n \times n$  matrix  $A(x) = (a_{ij}(x)) \in \mathbb{M}_{\mathbb{R}}^{n \times n}$  with  $a_{ij}(x) \in \mathbb{R}$  defines a linear map

$$T : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n.$$

We assume that  $a_{ij}(x)$  are bounded for all  $x \in \Omega : i, j = 1, 2, \dots, n$ .

Let

$$A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) & \dots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \dots & a_{2n}(x) \\ \dots & \dots & \dots & \dots \\ a_{n1}(x) & a_{n2}(x) & \dots & a_{nn}(x) \end{pmatrix}$$

We define the norm of  $A(x)$  as follows:

$$\begin{aligned} \|A(x)\| &= \max \left( \begin{array}{l} |a_{11}(x)| + |a_{12}(x)| + \dots + |a_{1n}(x)|, |a_{21}(x)| + |a_{22}(x)| + \dots + |a_{2n}(x)|, \dots, \\ |a_{n1}(x)| + |a_{n2}(x)| + \dots + |a_{nn}(x)| \end{array} \right) \\ &= \max \left( \sum_{j=1}^n |a_{1j}(x)|, \sum_{j=1}^n |a_{2j}(x)|, \dots, \sum_{j=1}^n |a_{nj}(x)| \right) \\ &= \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}(x)| \end{aligned}$$

Since  $a_{ij}(x); i, j = 1, 2, \dots, n$  are bounded, by assumption 1.2.3.1 (b), then is well-defined.

### 1.2.3.3. Existence and uniqueness of the solution to the problem:

There are several theorems that establish existence and uniqueness of solutions to initial value problems involving ODEs both locally and globally. A uniqueness theorem is the tool which makes it possible for us to conclude that there exists only one solution to a first order differential equation which satisfies a given initial condition. Furthermore,

existence and uniqueness solutions have been established under global Lipschitz conditions on the coefficients (Mohammed, 1984) or under local Lipschitz and linear growth conditions (Mao, 1997) and (Xu et al, 2008). In the study of differential equations, the Picard-Lindelof theorem, Picard's existence theorem or Cauchy-Lipschitz theorem and Peano existence theorem are important theorems on existence and uniqueness of solutions to first-order differential equations with given initial conditions. The Picard-Lindelof theorem shows that the solution exists and it is unique and also tells us that solutions to first order differential equations exist and unique under certain reasonable conditions. The Peano existence theorem shows only existence, not uniqueness, but it assumes only that  $f$  is continuous in  $y$ , instead of Lipschitz continuous. The Cauchy-Picard theorem states that if the right hand side of an ODE satisfies the Lipschitz conditions then boundary value problem has a unique solution. (Coddington & Levinson, 1955). One of the most important theorems in ordinary differential equations is Picard's existence and uniqueness theorem for first order ordinary differential equations. We will use the theorem to confirm the existence and uniqueness of the solutions to the problem at hand.

**Theorem 1.2.3.4: (Picard-Lindelof Theorem)**

We consider the following boundary value problem:

$$y'(x) = G(x, y), \quad y(x_0) = y_0, \quad x \in [x_0 - \epsilon, x_0 + \epsilon].$$

Suppose  $G$  is Lipschitz continuous on  $y$  and continuous on  $x$ . Then, for some arbitrary value  $\epsilon > 0$ , there exists a unique solution  $y(x)$  to the boundary value problem with

$$x \in [x_0 - \epsilon, x_0 + \epsilon].$$

Having stated above, the Picard - Lindelof Theorem above, we now apply it to the problem considered in the dissertation to prove an existence and uniqueness theorem.

### Theorem 1.2.3.5: (Main Theorem)

Let

$$y'(x) = G(x, y), \text{ where, } G(x, y) = A(x)Y(x) + b(x). \quad (4)$$

$$\text{Subject to : } Y(0) = \mathbf{c}$$

Then the solution to the problem is unique.

**Proof:**

For  $y$  to be a solution of (4)

$$y'(x) = G(\xi, y(\xi)).$$

Integrating both sides, we have

$$\int_0^x y'(x)dx = \int_0^x G(\xi, y(\xi))d\xi,$$

which implies that,

$$\begin{aligned} y(x) - y(0) &= \int_0^x G(\xi, y(\xi))d\xi \\ \Rightarrow y(x) &= \mathbf{c} + \int_0^x G(\xi, y(\xi))d\xi \end{aligned}$$

then,

$$\begin{aligned} \|G(x, y_1) - G(x, y_2)\| &= \|A(x)y_1(x) + b(x) - (A(x)y_2(x) + b(x))\| \\ &= \|A(x)y_1(x) + b(x) - A(x)y_2(x) - b(x)\| \\ &= \|A(x)y_1(x) - A(x)y_2(x)\| \end{aligned}$$

$$\|G(x, y_1) - G(x, y_2)\| \leq \|A(x)\|_{\mathbb{M}_{\mathbb{R}}^{n \times n}} \|y_1(x) - y_2(x)\|, \text{ implying that } G \text{ is Lipschitz on } y(x).$$

According to the Picard - Lindelof Theorem (Birkhoff & Rota, 1978), (Birkhoff & Rota, 1969) and (Coddington & Levinson, 1955) the solution  $y(x)$  is unique if and only if  $G(\xi, y(\xi))$  is Lipschitz on  $y$ . This clearly implies that  $G$  is Lipschitz continuous and the constant  $\|A(x)\|$  is called the Lipschitz constant of  $y(x)$  in  $\Omega$ . Since  $a_{ij}(x)$  are bounded by  $\|A(x)\|$  as defined in assumption 1.2.3.1.

## CHAPTER 2

### 2.1. The stability of the null solution to the problem:

We now define the stability of an equilibrium point. In general, the stability concerns the behavior of solutions near an equilibrium point in the long term (see **A4**). Equilibria are not always stable. Since stable and unstable equilibria play quite different roles in the dynamics of a system, it is useful to be able to classify equilibrium points based on their stability. In our situation, we will define the stability of the null solution through the Lefschetz direct stability criterion (Meyer, 1964). Firstly, we define the stability of matrix  $A$ , associated with (2).

#### Definition 2.1.1

Consider the following system :

$$z' = Az + b;$$

where  $A$  is an  $n \times n$  matrix with constant entries;

$$z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Let  $\lambda_j = \alpha_j \pm i\beta_j$ , be the eigenvalues of  $A$ , where  $\alpha_j, \beta_j \in \mathbb{R}; j = 1, 2, 3, \dots, n$ .

For the definition of stability we do not consider the imaginary part of  $\lambda_j$ . The following stability conditions hold for the real part  $\alpha_j$  :

If  $\alpha_j < 0$  for all  $j = 1, 2, \dots, n \Rightarrow A$  is stable,

$\alpha_j = 0$  for all  $j \Rightarrow A$  is saddle point or centre (it rotate either clockwise or anticlockwise)

If  $\alpha_j > 0 \Rightarrow A$  is hermitian matrix then its eigenvalues are real ( $\lambda_j \in \mathbb{R}$ ) always true. For at least are  $j$

(Hahn, 1967).

To study the stability of the solution to (2), we propose the direct method as presented by Lefschetz (Meyer, 1964).

However, to apply the Lefschetz direct stability criterion, we have to establish the stability of the matrix  $A(x)$ .

*Proof.*

We consider the Rayleigh quotient, where  $\lambda(x)$  is any eigenvalue of  $A(x)$  as  $x \in S$ . We define the Rayleigh quotient for  $A(x)$  as follows:

$$\lambda(x) = \frac{w^*(x) A(x) w(x)}{w^*(x) w(x)}, \quad (w(x) \text{ is a corresponding eigenvector to } \lambda(x))$$

Hence,

$$|\lambda(x)| = \frac{1}{|w(x)|^2} |w(x) A(x) w^*(x)|$$

The application of the Schwartz inequality and boundedness of  $A(x)$  leads to

$$|\lambda(x)| \leq \frac{\|A(x)\|_{\infty} \|w(x)\|_2}{\|w(x)\|_2} = \|A(x)\|_{\infty} = \max_{1 \leq j \leq n} |a_{jj}(x)|, \quad j = 1, 2, 3, \dots, n$$

$A(x)$ , for fixed  $x \in \mathbb{R}$  has only pure spectrum. That is hermitian matrix  $A(x)$  implies that the  $\lambda(x) \in \mathbb{R}^1$ . Hence,  $\lambda_{\max}(x)$  and  $\lambda(x) \leq \lambda_{\max}(x)$  for all  $x \in S$  and eigenvalue  $\lambda(x)$  of  $A(x)$ . Selecting:  $x \in D := \{x \in S : \lambda_{\max}(x) < 0\} \subset \mathbb{R}^1$ , then,  $A(x)$  is stable in  $D$ .

**Remarks 2.2.2:**

- (a) It can easily be shown that  $A(x)$  is not positively definite for all  $x \in \Omega$ .
- (b) Since  $A(x)$  is hermitian, by assumption 1.2.3.1 (e), then it is a continuous spectrum is real and hence, ordered.
- (c) In view of the preceding remarks, we can define  $D := \{x \in S : \lambda_{\max}(x) < 0\} \subset \mathbb{R}^1$ .

## 2.2. The stability of the matrix $A(x)$ :

We consider the spectrum for  $A(x)$  (see **A.3**) and then formulate the following Lemma.

**Lemma 2.2.1:** If  $A(x)$  is hermition matrix for each  $x \in A(x)$  is stable in  $D := \{x \in S : \lambda_{\max}(x) < 0\} \subset \mathbb{R}^1$ .

**Proof:**

We consider the Rayleigh quotient, where  $\lambda(x)$  is any eigenvalue of  $A(x)$  at  $x \in S$ .

We define the Rayleigh quotient for  $A(x)$  as follows:

$$\lambda(x) = \frac{\mathbf{w}^*(x) \cdot A(x) \mathbf{w}(x)}{\mathbf{w}^*(x) \cdot \mathbf{w}(x)}, \quad (\mathbf{w}(x) \text{ is a corresponding eigenvector to } (\lambda(x)))$$

Hence,

$$|\lambda(x)| = \frac{1}{\|\mathbf{w}(x)\|^2} |\mathbf{w}(x) \cdot A(x) \mathbf{w}^*(x)|.$$

The application of the Schwartz inequality and boundedness of  $A(x)$  leads to

$$|\lambda(x)| \leq \frac{\|A(x)\|_{M_{\mathbb{R}}^{n \times n}} \|\mathbf{w}(x)\|_{\mathbb{R}^n}}{\|\mathbf{w}(x)\|_{\mathbb{R}^n}} = \|A(x)\|_{M_{\mathbb{R}}^{n \times n}} = \max_{x \in \Omega} |a_{ij}(x)|; \quad i, j = 1, 2, 3, \dots, n$$

$A(x)$ , for fixed  $x \in \mathbb{R}$  has only part spectrum. That is hermition matrix  $A(x)$  implies that the  $\lambda(x) \in \mathbb{R}^1$ . Hence,  $\lambda_{\max}(x)$  and  $\lambda(x) \leq \lambda_{\max}(x)$  for all  $x \in S$  and eigenevalue  $\lambda(x)$  of  $A(x)$ . Selecting:  $x \in D := \{x \in S : \lambda_{\max}(x) < 0\} \subset \mathbb{R}^1$ , then,  $A(x)$  is stable in  $D$ .

**Remarks 2.2.2:**

- It can easily be shown that  $A(x)$  is not positively definite for all  $x \in \Omega$ .
- Since  $A(x)$  is hermition, by assumption 1.2.3.1 (a), then it is a continuous spectrum is real and hence, ordered.
- In view of the preceding remarks, we can define  $D := \{x \in S : \lambda_{\max}(x) < 0\} \subset \mathbb{R}^1$ .

## 2.3. The Lefschetz system for the stability analysis of the problem:

### 2.3.1. The original Lefschetz system:

We present the following Lefschetz system (see Meyer, 1964) on which the direct stability criterion would be based:

$$\left. \begin{aligned}
 x'(t) &= Ax(t) + b\phi(\sigma(t)), \\
 &\text{where,} \\
 \sigma(t) &:= c^T x(t); \text{ with the corresponding Lyapunov function:} \\
 V(x(t)) &:= (x(t))^T B(x(t)) + \int_0^{\sigma(t)} \phi(\sigma(t)) d\sigma(t); \\
 &\text{where } B \text{ is obtained from the Sylvester equation:} \\
 A^T B + BA &= -C; \quad -C \text{ arbitrary,} \\
 &\text{symmetric and positive definite.}
 \end{aligned} \right\} \quad (5)$$

### 2.3.2. The basic assumptions associated with this system are:

- (a)  $x, b, c$  are real  $n$  vectors; with  $n = 2$  in our case;
- (b)  $A$  is a real  $n \times n$  matrix with eigenvalues whose real parts are negative;
- (c)  $B$  is a positive definite symmetric matrix satisfying condition to be specified later;
- (d)  $\phi$  is a continuous function on  $\sigma$ ;
- (e)  $\sigma(0) = 0$ ; for  $\sigma \neq 0$ ,  $\sigma\phi(\sigma) > 0$ .

Provided  $A(x)$  is stable, it is confirmed (see Meyer, 1964), that  $-V(x(t)) > 0$  implies that

$$-c^T b > [Bb + \frac{1}{2}A^T c + \frac{1}{2}\alpha c^T]^T C^{-1} [Bb + \frac{1}{2}A^T c + \frac{1}{2}\alpha c^T]$$

Next, we present our modified Lefschetz system:

## 2.4. The modified Lefschetz system:

We consider the Lefschetz system (Meyer, 1964) and for the purpose of our study we modify it as follows:

$$\left. \begin{aligned} y'(x) &= A(x)y(x) + q\phi(\sigma(x)), \quad x \in D, \\ &\text{where,} \\ \sigma(x) &= c^T y(x); \quad c^T \text{ an } n \times 1 \text{ constant vector,} \\ \phi(\sigma(x)) &= \sqrt{\sigma(x)}, \quad \sigma(x) > 0; \text{ our arbitrary choice,} \\ &q : \text{ an } n \times 1 \text{ constant vector.} \end{aligned} \right\} \quad (6)$$

### Remarks 2.4.1

- (a)  $\phi$  is continuous on  $\sigma(x)$ , provided  $\sigma(x) > 0$  in  $D$ .
- (b) In terms of (2) and (5), we have  $q\phi(\sigma(x)) = b(x)$ .
- (c) The solution  $y(x)$  exists and is unique.
- (d) It is now clear that  $\sigma(x)\phi(\sigma(x)) > 0$ , which is one of the requirements by the Lefschetz direct stability criterion (Meyer, 1964).

In terms of the Lefschetz direct stability criterion, our Lyapunov function is defined as follows:

$$V(y(x)) := y^T(x)B(x)y(x) + \int_0^{\sigma(x)} \phi(\sigma(x))d\sigma(x), \quad \text{for } x \in D, \quad (7)$$

where, we will later show that  $B(x)$  is uniquely determined as the solution to the ODE:

$$\frac{dB(x)}{dx} + A^T(x)B(x) + B(x)A(x) = -C(x),$$

with  $-C(x)$  assumed positive definite and symmetric (see Makhabane & Hlomuka, 2011).

We will show that under certain conditions,  $B(x)$  is also positive definite.

### 2.4.2. Confirmation of the Lefschetz stability criterion using the modified system:

To differentiate a matrix with elements  $a_{ij} = a_{ij}(x)$ , we simply differentiate each element  $a_{ij}$  of that matrix with respect  $x$ .

From (7), we have,

$$\begin{aligned}
 \frac{d}{dx} [y^T(x)B(x)y(x)] &= \frac{dy^T(x)}{dx}B(x)y(x) + y^T(x)\frac{dB(x)}{dx}y(x) + y^T(x)B(x)\frac{dy(x)}{dx} \\
 &= (y^T)'(x)B(x)y(x) + y^T(x)B'(x)y(x) + y^T(x)B(x)y'(x) \\
 &= (y^T)'(x)B(x)y(x) + y^T(x)B(x)y'(x) + y^T(x)B'(x)y(x) \\
 &= 2B(x)y^T(x)y'(x) + y^T(x)B'(x)y(x)
 \end{aligned}$$

From the Lefschetz system, the Lyapunov function is

$$V = (y^T)'(x)B(x)y(x) + \int_0^\sigma \phi(\sigma(x))d\sigma(x)$$

Then,

$$\begin{aligned}
 V'(y(x)) &= 2B(x)y^T(x)y'(x) + y^T(x)\frac{dB(x)}{dx}y(x) + \int_0^{\sigma(x)} \phi(\sigma(x))d\sigma(x) \\
 &= 2B(x)y^T(x)y'(x) + y^T(x)\frac{dB(x)}{dx}y(x) + \frac{d}{dy} \int_0^{\sigma(x)} \phi(\sigma(x))d\sigma(x) \\
 &= 2B(x)y^T(x)y'(x) + y^T(x)\frac{dB(x)}{dx}y(x) + \phi(\sigma(x)) \left( \frac{d\sigma(x)}{dy(x)} \cdot \frac{dy(x)}{dx} \right) \\
 &= 2B(x)y^T(x)y'(x) + y^T(x)\frac{dB(x)}{dx}y(x) + \phi(\sigma(x)) \left( \frac{d\sigma(x)}{dy(x)} \cdot \frac{dy(x)}{dx} \right) \\
 &= 2B(x)y^T(x)y'(x) + y^T(x)\frac{dB(x)}{dx}y(x) + \phi(\sigma(x))c^T \cdot y'(x) \\
 &= 2y^T(x)B(x)y'(x) + y^T(x)\frac{dB(x)}{dx}y(x) + \phi(\sigma(x))c^T \cdot y'(x) \\
 &= 2y^T(x)B(x) [A(x)y(x) + q\phi(\sigma(x))] + y^T(x)\frac{dB(x)}{dx}y(x) + \phi(\sigma(x))c^T \\
 &\quad [A(x)y(x) + q\phi(\sigma(x))]
 \end{aligned}$$

That is,

$$\begin{aligned}
V'(y(x)) &= 2y^T(x)B(x)A(x)y(x) + 2y^T(x)B(x)q\phi(\sigma(x)) + y^T(x)\frac{dB(x)}{dx}y(x) + \\
&\quad \phi(\sigma(x))c^T A(x)y(x) + c^T q [\phi(\sigma(x))]^2 \\
&= y^T(x)A^T(x)B(x)y(x) + y^T(x)B(x)A(x)y(x) + y^T(x)\frac{dB(x)}{dx}y(x) + 2y^T(x)B(x)q \\
&\quad \phi(\sigma(x)) + \phi(\sigma(x))c^T A(x)y(x) + c^T q [\phi(\sigma(x))]^2 \\
&= y^T(x) [A^T(x)B(x) + B(x)A(x)] y(x) + y^T(x)\frac{dB(x)}{dx}y(x) + 2B(x)y(x)q\phi(\sigma(x)) + \\
&\quad \phi(\sigma(x))c^T A(x)y(x) + c^T q [\phi(\sigma(x))]^2 \\
&= y^T(x) (-C) y(x) + y^T(x)\frac{dB(x)}{dx}y(x) + 2 \left( B(x)q + \frac{1}{2}c^T A(x) \right) y(x)\phi(\sigma(x)) + \\
&\quad c^T q [\phi(\sigma(x))]^2 \\
V'(y(x)) &= -y^T(x)C y(x) + y^T(x)\frac{dB(x)}{dx}y(x) + 2 \left( B(x)q + \frac{1}{2}c^T A(x) \right) y(x)\phi(\sigma(x)) + \\
&\quad c^T q [\phi(\sigma(x))]^2
\end{aligned}$$

$$A^T(x)B(x)B(x)A(x) + \frac{dB(x)}{dx} = C(x); \quad (8)$$

with  $C(x)$  to be an arbitrary chosen (Makhabane & Hlomuka; 2011). Selecting  $\alpha > 0$ , we add and subtract  $\alpha\sigma\phi(\sigma(x)) > 0$  (as in Aizerman & Gantmacher; 1963):

Hence,

$$\begin{aligned}
-V'(y(x)) &= y^T(x)C(x)y(x) - 2 \left[ B(x)b + \frac{1}{2}A^T(x)c \right]^T y(x)\phi(\sigma) - c^T b\phi^2(\sigma) \\
&= y^T(x)C(x)y(x) - 2 \left[ B(x)b + \frac{1}{2}A^T(x)c \right]^T y(x)\phi(\sigma) - c^T b\phi^2(\sigma) + \alpha\sigma\phi(\sigma) - \alpha\sigma\phi(\sigma)
\end{aligned}$$

From the requirement that  $-V'(y(x))$  be positive definite, by using the Lefschetz stability criterion (Lyapunov, 1992) and (Massera, 1956).

$$\begin{aligned}
 V'(y(x)) &= y^T(x)C(x)y(x) - 2 \left[ B(x)q + \frac{1}{2}A^T(x)c \right]^T y(x)\phi(\sigma) - c^T q \phi^2(\sigma) + \alpha c^T y(x)\phi(\sigma) \\
 &\quad - \alpha c^T y(x)\phi(\sigma) \\
 &= y^T(x)C(x)y(x) - 2(B(x)q + \frac{1}{2}A^T(x)c + \frac{1}{2}\alpha c^T)y(x)\phi(\sigma) + \alpha c^T y\phi(\sigma) - c^T q \phi^2(\sigma)
 \end{aligned}$$

By using Lefschetz's condition we get

$$-V'(y(x)) \geq 0$$

Since  $\alpha c^T y(x)\phi(\sigma) > 0$ , then,

$$y^T(x)C(x)y(x) - 2 \left[ B(x)q + \frac{1}{2}A^T(x)c + \frac{1}{2}\alpha c^T \right] y(x)\phi(\sigma) - c^T q \phi^2(\sigma) \geq 0$$

Since the right hand side of above equation without  $\alpha \sigma \phi(\sigma(x)) > 0$  is positive quadratic expression in  $\phi(\sigma(x))$ , then  $b^2 - 4ac < 0$ , implies that,

$$\begin{aligned}
 4 \left[ B(x)q + \frac{1}{2}A^T(x)c + \frac{1}{2}\alpha c^T \right]^2 y^2(x) - 4 [-c^T q] [y^T(x)C(x)y(x)] &< 0 \\
 4 \left[ B(x)q + \frac{1}{2}A^T(x)c + \alpha c^T \right]^2 y^2(x) &< -4c^T q y^T(x)C(x)y(x) \\
 \frac{4 \left[ B(x)q + \frac{1}{2}A^T(x)c + \frac{1}{2}\alpha c^T \right]^2 y^2(x)}{4y^T(x)C(x)y(x)} &< \frac{-4c^T q y^T(x)C(x)y(x)}{4y^T(x)C(x)y(x)} \\
 \left[ B(x)q + \frac{1}{2}A^T(x)c + \frac{1}{2}\alpha c^T \right]^2 C^{-1}(x) &< -c^T q
 \end{aligned}$$

$$-c^T q > \left[ B(x)q + \frac{1}{2}A^T(x)c + \alpha c^T \right]^T C^{-1}(x) \left[ B(x)q + \frac{1}{2}A^T(x)c + \frac{1}{2}\alpha c^T \right] \quad (9)$$

like before.

## 2.5. The solution for $B(x)$ :

We solve for  $B(x)$  from the first order linear differential equation (6), with  $A(x)$  and  $B(x)$  assumed symmetric. The ODE (7) associated with our modified system (5) may be compared to the Sylvester equation in (5). The only difference is the term  $\frac{dB(x)}{dx}$ . We use the integrating factor  $e^{2xA(x)}$  to determine the analytical solution to the following equation, in view of (8):

$$-C(x) = A^T(x)B(x) + B(x)A(x) + \frac{dB(x)}{dx}.$$

We solve for  $B(x)$  using the above first order linear ordinary differential equation

$$-C(x) = y^T(x) [A^T(x)B(x) + B(x)A(x)] y(x) + y^T(x) \frac{dB(x)}{dx} y(x)$$

$$-C(x) = y^T(x) \left[ A^T(x)B(x) + B(x)A(x) + \frac{dB(x)}{dx} \right] y(x) \quad (10)$$

$$-C(x) = A^T(x)B(x) + B(x)A(x) + \frac{dB(x)}{dx}$$

$$-C(x) = A^T(x)B(x) + A(x)B(x) + \frac{dB(x)}{dx}$$

$$-C(x) = 2A(x)B(x) + \frac{dB(x)}{dx}$$

$$\frac{dB(x)}{dx} + 2A(x)B(x) = -C(x)$$

Multiplying throughout by  $e^{2xA(x)}$  to determine the analytical solution, we obtain

$$\frac{dB(x)}{dx} e^{2xA(x)} + 2A(x)B(x)e^{2xA(x)} = -C(x)e^{2xA(x)}$$

$$\frac{d}{dx} [B(x)e^{2xA(x)}] = -C(x)e^{2xA(x)} \quad (11)$$

$$\frac{d}{dx} [B(x)e^{2xA(x)}] = -e^{2xA(x)} C(x)$$

$$B(x)e^{2xA(x)} = -\int e^{2xA(x)} C(x) dx + k, \text{ where } k \text{ a constant}$$

$$B(x) = e^{-2xA(x)} \left[ -\int e^{2xA(x)} C(x) dx \right] + e^{-2Ax(x)} k$$

### 2.5.1 Positive definiteness and symmetric of $B(x)$ and $-C(x)$ :

We select  $-C(x) = e^{-2xA(x)}$

To find:

$$\begin{aligned}
 B(x) &= e^{-2xA(x)} \int e^{2xA(x)} (-C(x)) dx + e^{-2xA(x)} k, \text{ where, } k \geq 0. \\
 &= e^{-2xA(x)} \int e^{2xA(x)} (e^{-2xA(x)}) dx + e^{-2xA(x)} k \\
 &= e^{-2xA(x)} \int dx + e^{-2xA(x)} k \\
 &= e^{-2xA(x)} x + e^{-2xA(x)} k, \text{ } k \text{ a constant}
 \end{aligned}$$

Therefore,

$$B(x) = (x + k)e^{-2xA(x)}. \quad (10)$$

We impose the following conditions on  $e^{-2xA(x)}$ .

we put :  $0 \leq |x| \|A(x)\|_{M_{\mathbb{R}}^{n \times n}} < 1; x \in D$ .

This, in turn, implies that,  $0 \leq |x| |a_{ij}(x)| < 1$ .

Therefore,

$$0 \leq |x| \|A(x)\|_{M_{\mathbb{R}}^{n \times n}} < 1, \quad (11)$$

Implies that

$$\left. \begin{aligned}
 \text{(a) } 0 \leq -x |a_{ij}(x)| < 1 \\
 \text{or} \\
 \text{(b) } 0 \leq x |a_{ij}(x)| < 1
 \end{aligned} \right\} \quad (12)$$

From the exponential expansion, we have:

$$e^{-2xA(x)} = I - 2xA(x) + \frac{4x^2 A^2(x)}{2!} - \frac{8x^3 A^3(x)}{3!} + \frac{16x^4 A^4(x)}{4!} - \dots$$

By (11), we then define the following matrix exponential for  $A(x)$ :

$$\left. \begin{aligned} e^{-2xA(x)} &= I - 2xA(x) \\ e^{2xA(x)} &= I + xA(x) \end{aligned} \right\} \quad (13)$$

Finally, we have,

$$e^{-2xA(x)} = \begin{pmatrix} 1 - 2xa_{11}(x) & -2xa_{12}(x) & \cdots & -2xa_{1n}(x) \\ -2xa_{21}(x) & 1 - 2xa_{22}(x) & \cdots & -2xa_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ -2xa_{n1}(x) & -2xa_{n2}(x) & \cdots & 1 - 2xa_{nn}(x) \end{pmatrix}, \quad (14)$$

and

$$e^{2xA(x)} = \begin{pmatrix} 1 + 2xa_{11}(x) & 2xa_{12}(x) & \cdots & 2xa_{1n}(x) \\ 2xa_{21}(x) & 1 + 2xa_{22}(x) & \cdots & 2xa_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ 2xa_{n1}(x) & 2xa_{n2}(x) & \cdots & 1 + 2xa_{nn}(x) \end{pmatrix} \quad (15)$$

**Main theorem 2.5.2:**

For the inequality (10), and  $x \in D$ , the following are positive definite and symmetric:

- (a)  $e^{-2xA(x)}$ ;
- (b)  $-C(x)$ ;
- (c)  $B(x)$ .

**Proof:**

We wish to show that,

$$\mathbf{y}^T(x)e^{-2xA(x)}\mathbf{y}(x) \geq 0$$

$$\begin{aligned}
 &= \begin{pmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \end{pmatrix} \begin{pmatrix} 1 - 2xa_{11}(x) & -2xa_{12}(x) & \cdots & -2xa_{1n}(x) \\ -2xa_{21}(x) & 1 - 2xa_{22}(x) & \cdots & -2xa_{2n}(x) \\ \vdots & \vdots & \vdots & \vdots \\ -2xa_{n1}(x) & -2xa_{n2}(x) & \cdots & 1 - 2xa_{nn}(x) \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{pmatrix} \\
 &= \begin{pmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \end{pmatrix} \begin{pmatrix} y_1(x)(1 - 2xa_{11}(x)) - 2y_2(x)xa_{12}(x) + \cdots - 2y_n(x)xa_{1n}(x) \\ -2y_1(x)xa_{21}(x) + y_2(x)(1 - 2xa_{22}(x)) + \cdots - 2y_n(x)xa_{2n}(x) \\ \vdots \\ -2y_1(x)xa_{n1}(x) - 2y_2(x)xa_{n2}(x) + \cdots + y_n(x)(1 - xa_{nn}(x)) \end{pmatrix} \\
 &= y_1^2(x)(1 - xa_{11}(x)) - 2y_1(x)y_2(x)xa_{12}(x) + \cdots - 2y_1(x)y_n(x)xa_{1n}(x) - 2y_1(x)y_2(x)xa_{21}(x) \\
 &\quad + y_2^2(x)(1 - 2xa_{22}(x)) + \cdots - 2y_2(x)y_n(x)xa_{2n}(x) + \cdots - 2y_1(x)y_n(x)xa_{n1}(x) - 2y_2(x)y_n(x) \\
 &\quad xa_{n2}(x) + \cdots + y_n^2(x)(1 - 2xa_{nn}(x)) \\
 &\geq 0,
 \end{aligned}$$

if and only if 14 (a) holds.

The symmetric of  $e^{-2xA(x)}$  follows from the symmetric of  $A(x)$ .

(b) Having chosen  $-C(x) = e^{-2xA(x)}$ , the symmetric and positive definiteness of  $-C(x)$  follows.

(c) We have,  $B(x) = (x + k)e^{-2xA(x)}$ . Provided  $0 \leq x + k < 1$ ,  $B(x)$  is both symmetric and positive definite.

### 3.1. Construction of the analytical solution to the problem

Having studied the existence and uniqueness in chapter 2 and stability of the solution in chapter 3 we now set about constructing our analytical solution to the problem.

$$Y'(x) = A(x)Y(x) + b(x);$$

$$Y(0) = c$$

where

$$Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{pmatrix}; A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{pmatrix}; b(x) = \begin{pmatrix} b_1(x) \\ b_2(x) \\ \vdots \\ b_n(x) \end{pmatrix} \text{ and}$$

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

in an open bounded domain  $\Omega \subset \mathbb{R}^1$  with  $c$  as constant vector.

To solve the nonhomogeneous vector equation (2). Then we can write (2) as follows:

$$\frac{dY(x)}{dx} - A(x)Y(x) = b(x)$$

and multiply by the exponential matrix integrating factor  $e^{-Ax}$  to obtain

$$e^{-Ax} \frac{dY(x)}{dx} - e^{-Ax} A(x)Y(x) = e^{-Ax} b(x)$$

but by the product rule for differentiation of matrices, this is the same as

$$\frac{d}{dx} (e^{-Ax} Y(x)) = e^{-Ax} b(x)$$

## CHAPTER 3

### 3.1. Construction of the analytical solution to the problem:

Having studied the existence and uniqueness in chapter 2 and stability of its null solution in chapter 3 we now set about constructing our analytical solution to the problem.

Example:

$$Y'(x) = \mathbf{A}(x)y(x) + b(x);$$

$$y(0) = \mathbf{c}$$

where

$$Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{pmatrix}; \mathbf{A}(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{pmatrix}; b(x) = \begin{pmatrix} b_1(x) \\ b_2(x) \\ \vdots \\ b_n(x) \end{pmatrix} \text{ and}$$

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

in an open bounded domain  $\Omega \subset \mathbb{R}^1$ ; with  $\mathbf{c}$  as constant vector.

To solve the nonhomogeneous vector equation (2). Then we can write (2) as follows:

$$\frac{dy(x)}{dx} - A(x)y(x) = b(x)$$

and multiply by the exponential matrix integrating factor  $e^{-Ax}$  to obtain

$$e^{-Ax} \frac{dy(x)}{dx} - e^{-Ax} A(x)y(x) = e^{Ax} b(x)$$

but by the product rule for differentiation of matrices, this is the same as

$$\frac{d}{dx} (e^{-Ax} y(x)) = e^{-Ax} b(x).$$

Integration of both sides with constant vector  $\mathbf{c}$ , gives

$$e^{-Ax}y(x) = \int e^{-Ax}b(x)dx + \mathbf{c}$$

Recalling that  $e^{Ax}$  is the inverse of  $e^{-Ax}$ , we multiply by it to obtain the general solution

$$y(x) = e^{Ax} \int e^{-Ax}b(x)dx + e^{Ax}\mathbf{c} \quad (16)$$

**Example:**

Solve the following boundary values problem:

$$\frac{dy(x)}{dx} = \begin{pmatrix} 1 & x^2 \\ x^2 & 2x \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} + \begin{pmatrix} 1 \\ x \end{pmatrix}; \quad y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

**Solution:**

For any  $n \times n$  matrix  $A(x)$ ,

$$e^{A(x)} = I + A(x) + \frac{A^2(x)}{2!} + \frac{A^3(x)}{3!} + \dots$$

We impose the following conditions (11) and (12) to obtain:

$$\begin{aligned} e^{A(x)} &= I + A(x) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & x^2 \\ x^2 & 2x \end{pmatrix} \\ &= \begin{pmatrix} 2 & x^2 \\ x^2 & 1 + 2x \end{pmatrix} \end{aligned}$$

Similarly

$$\begin{aligned} e^{-A(x)} &= I - A(x) \\ &= \begin{pmatrix} 0 & -x^2 \\ -x^2 & 1 - 2x \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
 y(x) &= \begin{pmatrix} 2 & x^2 \\ x^2 & 1+2x \end{pmatrix} \int_0^x \begin{pmatrix} 0 & -x^2 \\ -x^2 & 1-2x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} dx + \begin{pmatrix} 2 & x^2 \\ x^2 & 2x+1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\
 &= \begin{pmatrix} 2 & x^2 \\ x^2 & 1+2x \end{pmatrix} \int_0^x \begin{pmatrix} -x^3 \\ x-3x^2 \end{pmatrix} dx + \begin{pmatrix} 2c_1 + x^2c_2 \\ x^2c_1 + 2xc_2 + c_2 \end{pmatrix} \\
 &= \begin{pmatrix} 2 & x^2 \\ x^2 & 1+2x \end{pmatrix} \begin{pmatrix} -\frac{1}{4}x^4 \\ \frac{1}{2}x^2 - x^3 \end{pmatrix} + \begin{pmatrix} 2c_1 + x^2c_2 \\ x^2c_1 + 2xc_2 + c_2 \end{pmatrix} \\
 &= \begin{pmatrix} -\frac{1}{2}x^4 + \frac{1}{2}x^4 - x^5 \\ -\frac{1}{4}x^6 + (1+2x)(\frac{1}{2}x^2 - x^3) \end{pmatrix} + \begin{pmatrix} 2c_1 + x^2c_2 \\ x^2c_1 + 2xc_2 + c_2 \end{pmatrix} \\
 &= \begin{pmatrix} -x^5 + 2c_1 + 2x^2c_2 \\ -\frac{1}{4}x^6 - 2x^4 + \frac{1}{2}x^2 + x^2c_1 + 2xc_2 + c_2 \end{pmatrix}.
 \end{aligned}$$

The general solution is then

$$y_1(x) = -x^5 + 2c_1 + 2x^2c_2$$

$$y_2(x) = -\frac{1}{4}x^6 - 2x^4 + \left(\frac{1}{2} + c_1\right)x^2 + 2xc_2 + c_2.$$

For  $y_1(0) = 1$

$$1 = -(0)^5 + 2c_1 + (0)^2c_2$$

$$1 = 2c_1$$

Therefore,

$$c_1 = \frac{1}{2}.$$

For  $y_2(0) = 1$

## CHAPTER 4

4.1. Summary

$$1 = -\frac{1}{4}(0)^6 - 2(0)^4 + \left(\frac{1}{2} + c_1\right)(0)^2 + 2(0)c_2 + c_2$$

$$1 = 0 + c_2.$$

Therefore,

$$c_2 = 1.$$

Our problem (2) has a general solution (15) which is global whilst stability is not.

Hence, our solution is

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} -x^5 + 2c_1 + 2x^2c_2 \\ -\frac{1}{4}x^6 - 2x^4 + \left(\frac{1}{2} + c_1\right)x^2 + 2xc_2 + c_2 \end{pmatrix}.$$

## CHAPTER 4.

### 4.1. Summary and conclusion:

We have employed successfully a condition for the existence of bounded solution on  $y(x)$  of the nonhomogeneous linear first order ordinary differential equation (2). The Picard-Lindelof Theorem was used to show that solutions to first order differential equation exist and are unique. The theorem (Picard-Lindelof) successfully confirmed that solution  $y(x)$  is unique if and only if  $G(\xi, y(\xi))$  is Lipschitz on  $y(x)$ . The investigations were conducted to show linear stability of the matrix  $A(x)$  and the analysis tells us how a system behaves near an equilibrium point.

In several cases it is easier to confirm asymptotic stability than stability. The method whereby stability is studied without asymptotic stability is referred to as a direct stability method. We have chosen the Lefschetz direct stability method and modified it to suit our problem at hand. The direct method required the construction of a Lyapunov function; not easy for a time independent problem. For a dynamic problem the energy thereof is a suitable candidate for a Lyapunov function. For a time-independent (non-dynamic) problem it is harder to construct a Lyapunov function as there is no rules for the purpose. In our study we modified the Lefschetz system for the direct stability method and applied it to study the stability of a system of linear first order ODEs with variable coefficients. We observed that Main Theorem 2.5.2 validates our choices of the Lyapunov function (6) and the solution for  $B(x)$  (9) and hence, the criterion (8). What is remarkable is that a criterion based on a dynamic Lefschetz system could still be applied to a non-dynamic system (2), thus afford us the confirmation of Lefschetz direct stability system. We solved the Makhabane / Hlomuka equation (7) for  $B(x)$  by Sylvester's equation which we later confirmed as both symmetric and positive definite.

## Appendix:

### A.1: Preliminaries:

#### A.1.1: Boundary and initial value problems:

**Definition A.1.1.1:** An initial value problem is a differential equation

$$y'(x) = G(x, y) \quad \text{with} \quad T : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \text{where } T \text{ is a linear map,}$$

together with a point in the domain of  $T$

$$(x_0, y_0) \in \Omega,$$

called the initial condition.

For our existence and uniqueness study, we need the Picard-Lindelof Theorem:

Before we stated the Theorem, we need the following definition.

**Definition A.1.1.2:** A function  $G(x, y)$  is called Lipschitz in  $y$  in  $\Omega$  if there is a constant  $L > 0$  such that for all  $(x, y_1), (x, y_2) \in \Omega$

$$\|G(x, y_1) - G(x, y_2)\| \leq L \|y_1 - y_2\|.$$

In the view of the equivalence of any two norms in  $\mathbb{R}^1$ , the Lipschitz condition does not depend on the choice of the norm (but the value of the Lipschitz constant  $L$  does).

### A.2: Matrix norms:

We confine ourselves to square matrices. A complex matrix  $P$  of order  $n$  can be identified as an element of  $\mathbb{M}_{\mathbb{R}}^{n \times n}$ . However, since the space matrices  $\mathbb{M}_{\mathbb{R}}^{n \times n}$  is a vector space it can be equipped with a norm. These matrix norm give insight into how sensitive the system is to rounding-off errors. However most properties of vector norms

are retained and nothing prevents us from using them to measure the size, and the distance between, matrices. For a corehere matrix  $P = (p_{ij})$ , a matrix norm  $\|P\|$  is a nonnegative number associated with  $P$ .

**Definition A.2.1:** A matrix norm is a function  $\|\cdot\| : \mathbb{M}_{\mathbb{R}}^{n \times n} \subset \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$  which, for all  $P, Q \in \mathbb{M}_{\mathbb{R}}^{n \times n}$  and all  $\alpha \in \mathbb{R}$ , satisfies the following properties:

(a)  $\|P\| \geq 0$ , and  $\|P\| = 0 \Leftrightarrow P = 0$ , ,

(b)  $\|\alpha P\| = |\alpha| \|P\|$

(c)  $\|PQ\| \leq \|P\| \|Q\|$

(d)  $\|P + Q\| \leq \|P\| + \|Q\|$

**Definition A.2.2:** Given a vector norm  $\|\cdot\|_{\mathbb{R}}$  on  $\mathbb{R}^1$  we define the induced norm  $\|\cdot\|_{\mathbb{R}}$  on  $\mathbb{M}_{\mathbb{R}}^{n \times n} \subset \mathbb{R}^1$  by

$$\|P\|_{\mathbb{M}_{\mathbb{R}}^{n \times n}} = \max_{x \neq 0} \frac{\|Px\|_{\mathbb{R}}}{\|x\|_{\mathbb{R}}}$$

for all  $P \in \mathbb{M}_{\mathbb{R}}^{n \times n} \subset \mathbb{R}^1$  (Bhatia, 1997).

We now show that the induced norm is indeed a norm. The induced norm  $\|\cdot\|_{\mathbb{R}^n}$  of a vector norm  $\|\cdot\|_{\mathbb{R}^n}$  is a matrix norm with  $\|I\|_{\mathbb{R}^n} = 1$  and  $\|Px\|_{\mathbb{M}_{\mathbb{R}}^{n \times n}} \leq \|P\|_{\mathbb{M}_{\mathbb{R}}^{n \times n}} \|x\|_{\mathbb{R}^n}$  for all  $P \in \mathbb{M}_{\mathbb{R}}^{n \times n}$ . Conditions (a), (b) and (c) state that  $\|\cdot\|$  is a vector norm on the vector space  $\mathbb{M}_{\mathbb{R}}^{n \times n} \subset \mathbb{R}^1$ . Condition (c) only makes sense for matrices, since general vectors spaces are not equipped with a product. Let's prove the properties one-by-one. Many of the proofs simply rely on the properties satisfies by the Euclidean norm:

**Proof:**

(a)  $\|P\|_{\mathbb{M}_{\mathbb{R}}^{n \times n}} \in \mathbb{R}$  and  $\|P\|_{\mathbb{M}_{\mathbb{R}}^{n \times n}} \geq 0$  for all  $P \in \mathbb{M}_{\mathbb{R}}^{n \times n}$  from the definition. Also from the definition we get

$$\begin{aligned}
 \|P\|_{\mathbb{M}_{\mathbb{R}^n}^{n \times n}} &= 0 \\
 &\Leftrightarrow \frac{\|Px\|_{\mathbb{R}^n}}{\|x\|_{\mathbb{R}^n}} = 0 \quad \forall x \neq 0 \\
 &\Leftrightarrow \|Px\|_{\mathbb{R}^n} = 0 \quad \forall x \neq 0 \\
 &\Leftrightarrow Px = 0 \quad \forall x \neq 0 \\
 P &= 0.
 \end{aligned}$$

(b) For  $\alpha \in \mathbb{R}$  and  $P \in \mathbb{M}_{\mathbb{R}^n}^{n \times n}$  we get

$$\begin{aligned}
 \|\alpha P\|_{\mathbb{M}_{\mathbb{R}^n}^{n \times n}} &= \max_{x \neq 0} \frac{\|\alpha Px\|_{\mathbb{R}^n}}{\|x\|_{\mathbb{R}^n}} \\
 &= \max_{x \neq 0} \frac{|\alpha| \|Px\|_{\mathbb{R}^n}}{\|x\|_{\mathbb{R}^n}} \\
 &= |\alpha| \|P\|_{\mathbb{M}_{\mathbb{R}^n}^{n \times n}}.
 \end{aligned}$$

Before we check condition (c) from the definition of a matrix norm we verify

$$\begin{aligned}
 \|I\|_{\mathbb{M}_{\mathbb{R}^n}^{n \times n}} &= \max_{x \neq 0} \frac{\|Ix\|_{\mathbb{R}^n}}{\|x\|_{\mathbb{R}^n}} \\
 &= \max_{x \neq 0} \frac{\|x\|_{\mathbb{R}^n}}{\|x\|_{\mathbb{R}^n}} \\
 &= 1,
 \end{aligned}$$

and

$$\|P\|_{\mathbb{M}_{\mathbb{R}^n}^{n \times n}} = \max_{y \neq 0} \frac{\|Py\|_{\mathbb{R}^n}}{\|y\|_{\mathbb{R}^n}} \geq \frac{\|Px\|_{\mathbb{R}^n}}{\|x\|_{\mathbb{R}^n}} \quad \forall x \in \mathbb{R}^n \setminus \{0\}$$

which gives

$$\|Px\|_{\mathbb{R}^n} \leq \|P\|_{\mathbb{M}_{\mathbb{R}^n}^{n \times n}} \|x\|_{\mathbb{R}^n} \quad \forall x \in \mathbb{R}^n.$$

(c) Using this estimate we find

### A.3: The Spectrum for the Matrix $A(x)$ :

Let  $\lambda(x)$ ,  $x \in \Omega$  be the eigenvalue of  $A(x)$  for some non-zero vector  $y(x)$ . Then the set  $\sigma(A(x))$  will denote the spectrum of a matrix  $A(x)$ . That is  $\sigma(A(x)) = \{\lambda(x) : A(x)y(x) = \lambda(x)y(x)\}$  an eigenvector of a matrix  $A$ , then it is easy to determine the value of  $\lambda(x)$  for any  $x$ . For this purpose,

$$\begin{aligned} \|PQ\|_{\mathbb{M}_{\mathbb{R}^n}^{n \times n}} &= \max_{x \neq 0} \frac{\|PQx\|_{\mathbb{R}^n}}{\|x\|_{\mathbb{R}^n}} \\ &\leq \max_{x \neq 0} \frac{\|P\|_m \|Qx\|_{\mathbb{R}^n}}{\|x\|_{\mathbb{R}^n}} \\ &= \|P\|_{\mathbb{M}_{\mathbb{R}^n}^{n \times n}} \|Q\|_{\mathbb{M}_{\mathbb{R}^n}^{n \times n}} \end{aligned}$$

1. (d) For  $P, Q \in \mathbb{M}_{\mathbb{R}^n}^{n \times n} \subset \mathbb{R}^1$  we get

$$\begin{aligned} \|P + Q\|_{\mathbb{M}_{\mathbb{R}^n}^{n \times n}} &= \max_{x \neq 0} \frac{\|Px + Qx\|_{\mathbb{R}^n}}{\|x\|_{\mathbb{R}^n}} \\ &\leq \max_{x \neq 0} \frac{\|Px\|_{\mathbb{R}^n} + \|Qx\|_{\mathbb{R}^n}}{\|x\|_{\mathbb{R}^n}} \\ &\leq \max_{x \neq 0} \frac{\|Px\|_{\mathbb{R}^n}}{\|x\|_{\mathbb{R}^n}} + \max_{x \neq 0} \frac{\|Qx\|_{\mathbb{R}^n}}{\|x\|_{\mathbb{R}^n}} \\ &= \|P\|_{\mathbb{M}_{\mathbb{R}^n}^{n \times n}} + \|Q\|_{\mathbb{M}_{\mathbb{R}^n}^{n \times n}}. \end{aligned}$$

(Trefethen & Bau, 1997) and (Cormen et al, 2001).

We say that the matrix norm  $\|\cdot\|$  is compatible with the vector norm  $\|\cdot\|'$  provided  $\|Px\|' \leq \|P\| \|x\|'$  holds for every matrix  $P$  of order  $n$  and every  $n$ -vector  $x$ . Where do matrix norms come from? Perhaps the most natural answer can be given along the following lines. Consider the scalar function  $f(x) = Px$ , where  $P$  and  $x$  are complex. The number  $|P|$  gives the amount by which the argument  $x$  is stretched by the function  $f$ , since  $|f(x)| = |P| |x|$ . In the more general situation where  $P$  is replaced by the matrix  $P$  and  $x$  is a vector, we cannot expect  $P$  to stretch all vectors by the same amount. In other words, there may not be a constant  $c$  such that  $\|Px\| = c \|x\|$  for all  $x$ . However, there is a smallest constant  $c$  such that  $\|Px\| \leq c \|x\|$ . The constant  $c$  is called the norm of  $P$  subordinate to the vector norm  $\|\cdot\|$ . We define the matrix norm subordinate to the vector norm  $\|\cdot\|$  by

$$\|P\| = \sup_{x \neq 0} \frac{\|Px\|}{\|x\|}$$

(Horn & Johnson, 1985) and (Chow & Hale, 1996).

### A.3: The Spectrum for the Matrix $A(x)$ :

Let  $\lambda(x)$ ,  $x \in \Omega$  be the such that,  $A(x)y(x) = \lambda(x)y(x)$  for some non-zero vector  $y(x)$ . Then the set  $\sigma(A(x))$  will denote the spectrum for the matrix  $A(x)$ . That is:

$\sigma(A(x)) = \{\lambda(x) : A(x)y(x) = \lambda(x)y(x)\}$ . If a vector  $w$  is an eigenvector of a matrix  $A$ , then it is easy to determine the value of the associated eigenvalue. For this purpose, we use the Rayleigh quotient of the matrix  $A \in \mathbb{R}^{n \times n}$  with respect to the vector  $w \in \mathbb{R}^n$  is

$$\lambda(x) = \frac{\mathbf{w}^T A \mathbf{w}}{\mathbf{w}^T \mathbf{w}} \in \mathbb{R}.$$

For a general non-zero vector  $w$  lies between the smallest and largest eigenvalues of the matrix  $A$ . If the matrix  $A$  is symmetric, then all eigenvalues are real, and we may write the simpler statement:

$$\lambda(x)_{\min} \leq \lambda(x) \leq \lambda(x)_{\max}$$

When  $w$  is an approximate eigenvector of  $A$ , the Rayleigh quotient is an accurate estimate of the corresponding eigenvalue (if real) or the magnitude of the eigenvalue (Horn & Johnson, 1985).

### A.4: The Lyapunov stability theory:

Linear stability analyses tell us how a system behaves near an equilibrium point. It does not however tell us anything about what happens further away from equilibrium. Phase-plane analysis combined with linear stability analysis can generally give us a full picture of the dynamics, but things become much more difficult in higher-dimensional spaces. In this section, we consider a technique due to Lyapunov which can be used to determine the stability of an equilibrium point “in the large”, i.e. both near and far from the equilibrium point. Lyapunov’s method is based on a simple idea. Suppose that  $V(x)$  is a function of our state variables which has a minimum at an equilibrium

point and which has no local minima. Now suppose that we can show that the dynamics of our system results in a steady decrease in  $V$  in some (possible large) neighborhood of the equilibrium point. this necessary means that we are tending toward the minimum of  $V$ , which is just the equilibrium point. Having shown this we conclude that the equilibrium point is stable over the entire neighborhood of  $x^*$  over which  $V$  decreases. A function  $V$  with these properties is called a Lyapunov function.

Let us formalize this idea in mathematically as follows:

Consider the linear ODE

$$\Rightarrow \dot{x} = f(t, x) \quad x(t_0) = x_0 \in \mathbb{R}^n$$

Here  $f : \Omega \times \mathbb{R}^1 \rightarrow \mathbb{R}^1$  be a given continuous function. We will assume that the system given above has a unique solution for any initial condition  $x_0$ .  $x = x^*$  is called an equilibrium point if and only if  $f(t, x^*) = 0$  for all  $t$ . Interpretation:  $x(t_0) = x^* \Rightarrow x(t) = x^*$  for all  $t$ . It is because the solution  $x(t) = x^*$  satisfies the ODE and is unique (Lyapunov; 1992).

Lyapunov stability is related to the stability of this solution  $x(t) = x^*$ , hence it is related to the equilibrium points. What happens to this solution if  $x(t_0) \neq x^*$ , but is close, i.e.  $\|x(t_0) - x^*\|$  is small? Intuitively, if  $\|x(t) - x^*\|$  also remains “small”, we will call  $x^*$  as a stable equilibrium. If  $\|x(t) - x^*\|$  becomes “large” (not necessarily diverging to  $\infty$ ), we will call  $x^*$  as an unstable equilibrium. If  $\|x(t) - x^*\| \rightarrow 0$ , we will call  $x^*$  as an *asymptotically stable* equilibrium. In these cases, we mention “The stability of  $x^*$  in the sense of Lyapunov”. If  $x^* \neq 0$ , it is customary to shift it to  $0 \Rightarrow \dot{z} = \dot{x} - \dot{x}^*$ .

$\Rightarrow z = x - x^* = f(t, x) - f(t, x^*) = F(t, z)$ .  $f(t, x^*) = 0 \Leftrightarrow F(t, 0) = 0$ . Hence, without loss of generality, we will assume that  $x^* = 0$ . (All the following definitions / results will be valid if we replace  $x$  by  $x - x^*$ ) (Hahn, 1967) and (Massera, 1949).

**Definition:** The equilibrium point  $x^* = 0$  is called *stable* if for any  $\varepsilon > 0$ , we can find  $\delta(\varepsilon, t_0) > 0$  such that

## A.5: The method of solving problem (2):

$$\|x(t_0)\| \leq \delta \Rightarrow \|x(t)\| \leq \varepsilon \quad \forall t \geq t_0$$

**Definition:** The equilibrium point  $x^* = 0$  is called *uniformly stable* if it is stable and  $\delta = \delta(\varepsilon)$  (i.e. does not depend to  $t_0$ )

**Definition:** The equilibrium point  $x^* = 0$  is called *unstable* if it is not stable.

**Definition:** ((sign) definite function) A function  $f : \Omega \times \mathbb{R}^1 \rightarrow \mathbb{R}^1$  is positive definite at  $x^* \in \Omega \times \mathbb{R}^1$  if

(i)  $f(x^*) = 0$  and

(ii)  $f(x) > 0$  for all  $x \in \Omega \times \mathbb{R}^1$  with  $x \neq x^*$ .

$f$  is negative definite if  $-f$  is positive definite.

**Definition:** (Lyapunov function) A continuously differentiable function

$f : \Omega \times \mathbb{R}^1 \rightarrow \mathbb{R}^1$ , where  $\Omega \times \mathbb{R}^1 \subset \mathbb{R}^1$ , is a Lyapunov function for  $x' = f(x)$  at  $x^* \in \Omega \times \mathbb{R}^1$  if

(i)  $f(x)$  is positive definite at  $x^*$ , and

(ii)  $f'(x) \leq 0$  for all  $x \in \Omega \times \mathbb{R}^1$

If in addition  $f'(x)$  is negative definite at  $x^*$ , then  $f$  is a strict Lyapunov function

$$x(t_0) \in B_\delta \Rightarrow \|x(t)\| \rightarrow 0.$$

**Definition:** The equilibrium point  $x^* = 0$  is called *globally asymptotically stable* if it is stable and  $\|x(t)\| \rightarrow 0 \quad \forall x(t_0) \in \mathbb{R}^1$ . Means  $\delta = \infty$ , i.e.  $B_\delta = \mathbb{R}^1$ .

**Definition:** The equilibrium point  $x^* = 0$  is called *locally exponentially stable* if  $\exists$  positive constants  $r, M, \delta$  such that

$$\|x(t_0)\| \leq r \Rightarrow \|x(t)\| \leq M e^{-\delta(t-t_0)} \|x(t_0)\|$$

and is called *globally exponentially stable* if  $r = \infty$  (i.e. above inequality holds  $\forall (t_0) \in \mathbb{R}^1$ ) (Massera, 1956) and (Malkin, 1977).

## A.5: The method of solving problem (2):

In this study, several methods are used to investigate the existence and uniqueness; the solution and its behaviour (i.e. stability). The methods are as follows:

(i) Lipschitz conditions (for investigating existence and uniqueness).

(a) Suppose that  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ , then  $f$  is Lipschitz conditions on the open subset  $\Omega$  iff  $\Lambda \in \mathbb{R}_0^+$  such that

$$\|f(x) - f(y)\| \leq \Lambda \|x - y\|, \quad \forall x, y \in \Omega.$$

(b) If for each  $z \in \mathbb{R}^n$  there exists  $L > 0$  such that  $f$  is Lipschitz conditions on the open ball of center  $z$  and radius  $L$ , and  $f$  is called locally Lipschitz continuous.

(c) If  $f$  is a Lipschitz - continuous on all of the space  $\mathbb{R}^n$ , then  $f$  is called globally Lipschitz continuous.

(ii) The Lyapunov theory will help us to analyses the stability of our problem in a way that the non-linear first ODEs, variable coefficients

$$\Rightarrow x' = f(t, x), \quad x(t_0 = x_0 \in \mathbb{R}^n).$$

(iii) The modification of the original Lefschetz system to conform the Lefschetz stability criterion using modified system.

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