



University of Venda

A Masters Dissertation

Existence and Uniqueness of a solution to a flow problem about a
Rotating Obstacle at low Reynolds number

By

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Symbol Explanation

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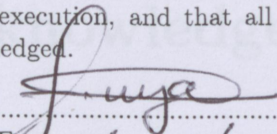
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Symbol Explanation

- \vec{x} – (x, y, z) spatial coordinate in Ω ;
- $L^2(\Omega)$ – space of the Lebesgue integrable functions on Ω ;
- $H^2(\Omega)$ – the Sobolev space of order 2 on Ω ;
- $\vec{u}(\vec{x}, t)$ – the velocity field of the flow in Ω ;
- $p(\vec{x}, t)$ – the fluid pressure;
- $\vec{\omega}$ – angular velocity of the rotating obstacle, assumed constant;
- ρ – fluid mass density, assumed constant;
- μ – coefficient of viscosity, assumed constant;
- $\vec{n}(\vec{x})$ – outward unit vector to Γ ;
- $\gamma_0 \vec{u}(\vec{x}, t)$ – velocity at $\vec{x} \in \Gamma$;
- $\eta_u(\vec{x}, t)$ – the normal velocity components at $\vec{x} \in \Gamma$;
- $\gamma_0 \vec{u}(\vec{x}, t)$ – 0, at the stagnation point;
- $\nabla \vec{u}_{i,j}$ – the velocity gradient;
- $D(\vec{u})$ – the rate of deformation tensor;
- Θ – set of test functions in Ω ;
- τ – Represents thickness of the fluid rotating with a circular cylinder(\mathbb{I})

Declaration

I declare that the dissertation hereby submitted to the University of Venda for the award of a Masters of Science (MSc) degree in Mathematics has not been previously submitted elsewhere, that it is my own work in design and in execution, and that all material contained therein has been duly acknowledged.

Signed.....

Nyathi Freeman

Date.....12/04/2016

Signed.....

Supervisor

Date.....18/04/2016

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Chapter 1

Introduction

Abstract

The flow is described by the Navier-Stokes equations in the domain $\Omega \subset R^3$. The open-bounded domain is assumed to have a cone property. The rotation of a 3- dimensional symmetrical impermeable cylindrical rigid body in the fluid is studied. The model is constructed in a manner that the equations describe a system in a frame attached to the obstacle. The system of the governing equations is constructed on the basis of conservation of angular momentum of the rigid body and the conservation of linear momentum of the fluid. When the conservation of angular momentum is taken into account, a dynamic boundary condition is considered. The uniqueness of this unknown velocity vector field is confirmed by using the so called energy method. In this study we chose the incompressible viscous Navier-Stokes flow and thus the fluid density does not change through out the flow.

Chapter 1

Introduction

1.1 Introduction

Flows around obstacles are among the most common types encountered in the fluid mechanics literature, and cylindrical obstacles have received particularly broad attention. The reason is that, this relatively simple geometry encompasses many important physical effects likely to play a role in the flow around complicated obstacles. This implies that a profound understanding of the flow past a cylinder will facilitate the analysis of a wide variety of applied problems, ranging from aerodynamic to pollutant dispersion around buildings and flows in turbines.

The study will adopt the equations of a Navier-Stokes fluid, which were derived a long time ago and the term that will be added is $[(\vec{\omega} \times \vec{x}) \cdot \nabla] \vec{u}$ due to the drift operator of the rotating body. i.e. the body in a fluid flow will oscillate about a fixed position. A Navier-Stokes fluid is an incompressible Newtonian fluid. When the fluid is not a mixture, incompressibility means a constant fluid density. We shall consider only such fluids.

Newtonian fluids are characterised by the linear occurrence of the rate of deformation in the stress tensor, a term which contains the viscosity of the fluid as a physical constant. Viscous fluids are often modelled as having the property of embedding to a boundary of a rotating obstacle and rotate with it.

We deal with functions in the spaces, $L^2([0, T])$ and $H^2(\Omega)$ respectively. In these spaces, we select appropriate test functions with compact support in the open bounded domain Ω . Those test functions are candidates for a weak solution of the problem (in the sense of distributions).

A model for studying this situation has been studied for its local existence

and uniqueness in ([6] and [8]) where both authors considered the effects of pressure forces in the fluid as well as effects of pressure forces at large distances from the body and, in this project, we confirm the same existence and uniqueness of the solution of the problem using a different method.i.e. energy method, where we are only considering a body rotating in a fluid without the effects of pressure forces at a distances away from the body.

The energy is a powerful method when dealing with problem in fluid mechanics. When applying the method in this problem, it allows us to go to an extend of showing the exercise in chapter 4 in the uniqueness of the problem.

1.2 Incompressible Navier-Stokes flow

The problem is set as a 3-dimensional viscous, incompressible fluid flow around a smooth impenetrable regular obstruction with a fixed axis of rotation in the vertical direction. The resulting motion of the fluid in the presence of a rotating obstacle about the z -axis is rotational.i.e. the body rotates about a fixed axis, z , at an angular velocity $\vec{\omega} = (0, 0, z)$ which is assumed to be sufficiently small. When the fluid is in contact with the rotating obstacle, the normal velocity component of the fluid velocity on the surface is zero. All the other surface velocity components are tangential to the rotating obstacle.This velocity at the boundary, given by $\gamma_0 \vec{u}(\vec{x}, t)$, for $\vec{x} \in \Gamma$, is prescribed in terms of no-slip condition of the fluid in contact with a rotating obstacle. Flow particles adjacent to the body tend to stick to it and eventually rotate with it.

In this project we look at the flow dynamics of a rotating body in an unbounded fluid. When this obstacle starts to rotate, there will be some boundary layer that will be formed on it. The Oseen [19] approach would mean that we have to deal with two problems.i.e. the boundary layer problem and the outer one. This was pointed out by Stokes, who gave the following explanation.

The pressure of the cylinder on the fluid continually tends to increase the quantity of fluid which it carries with it, while the friction of the fluid at a distance from the cylinder continually tends to diminish it. In the case of a sphere these two causes eventually counteract each other and the motion becomes uniform. But in case, of a cylinder the increase in the quantity of fluid carried continually gains on the decrease due to the friction of the surrounding fluid, and the quantity carried increases indefinitely as the cylinder moves on.

Chapter 2

Literature review

Fluid motion as a result of a rotating obstacle is one of the most fascinating problems in fluid dynamics. Research in this area has followed several avenues. Early work by Yoshihiro Shibata [23] came up with the stability theorem of three dimensional Navier-Stokes flow past a rotating rigid body. He considered the Navier-Stokes equations describing a viscous incompressible flow past a rotating rigid body in the Euclidean space R^3 .

In order to understand why the problem becomes difficult if the obstacle was rotating and translating, we recall the method typically employed in the study of the asymptotic structure of the steady state solution in exterior domain (Galdi, [6] and [7]) relies upon the proof of existence and of appropriate estimates of solutions to the *linearized* problem, in conjunction with a suitable fixed point argument. In turn, this proof was typically achieved by showing appropriate estimate of the fundamental solution for the relevant linear operator. Now, if the obstacle was only translating, the linearized operator was the well-known *Oseen operator*, Onseen [19], L_T , which was obtained from the (second order) Stokes operator by adding a lower (first) order term in the velocity field \vec{u} , with constant coefficients. If, on the other hand, the body was rotating with angular velocity $\vec{\omega}$, the corresponding linearized operator, L_{TR} , also includes the first order term $(\vec{\omega} \times \vec{x}) \cdot \nabla$. This term becomes unbounded at large distances from the body.

Galdi and Silvestre [6], [7] proved uniqueness of the steady state problem:

$$-\Delta \vec{u} - (\vec{\omega} \times \vec{x}) \cdot \nabla \vec{u} + \vec{\omega} \times \vec{u} + \nabla p + \vec{u} \cdot \nabla \vec{u} = 0, \nabla \cdot \vec{u} = 0,$$

defined in D with the solution (\vec{u}, p) satisfying the inequality:

$$|\vec{u}(\vec{x})| \leq \frac{C|\vec{\omega}|}{|\vec{x}|}, |\nabla \vec{u}(\vec{x})| + |p(\vec{x})| \leq \frac{C|\vec{\omega}|^2}{|\vec{x}|},$$

for large $|\vec{x}|$ provided $|\vec{\omega}|$ is small enough.

Hishida [8], proved pointwise decay of:

$$|\vec{u}(\vec{x})| \leq \frac{C|\vec{\omega}|}{|\vec{x}|}, |\nabla\vec{u}(\vec{x})| + |p(\vec{x})| \leq \frac{C|\vec{\omega}|^2}{|\vec{x}|^2},$$

in terms of *weak* - L_q spaces:

$$\vec{u} \in L_{3,\infty}(D), (\nabla\vec{u}, p) \in L_{3/2,\infty}(D).$$

The general equations of determining a solution to a problem of a flow past an obstacle is the general Navier-Stokes equation. However if the body rotates and translates the linearized problem becomes, depending on specially chosen initial data:

$$\partial_t\vec{u} + \vec{u} \cdot \nabla\vec{u} = \Delta\vec{u} + (\vec{\omega} \times \vec{x}) \cdot \nabla\vec{u} - \vec{\omega} \times \vec{u} - \nabla p, \nabla \cdot \vec{u} = 0,$$

subject to the conditions :

$$\vec{u}_\Gamma = \vec{\omega} \times \vec{x}, \vec{u} \rightarrow 0 \text{ point-wise as } |\vec{x}| \rightarrow \infty \text{ (Hishida [8]).}$$

If the body was just rotating without translating the term that will be added is:

$$[(\vec{\omega} \times \vec{x}) \cdot \nabla]\vec{u},$$

due to the velocity of the rotating body(see Galdi, [6] and [7]). Stokes [24], solved, in two dimensions, a problem of a flow past a circular cylinder of radius a . The linearized equation is:

$$\mu\nabla^2\vec{\omega} = \mu\nabla^4\psi = 0.$$

The no-slip condition at the surface of the cylinder requires that:

$$\psi(a) = \frac{\partial\psi(a)}{\partial r} = 0.$$

There were no global solutions to the 2-dimensional Stokes problem representing a flow past a cylinder. This phenomenon results from a failure to properly account for the forces in a viscous fluid at large distance from a translating body and is called the Stokes Paradox. The remedy for this phenomenon in two dimensions involves a proper accounting for singular nature of the limit $\text{Re} \rightarrow 0$ in the neighbourhood of infinity. At distance

$$r \rightarrow \text{Re}^{-1},$$

the appropriate equations are found to be:

$$\rho \frac{\partial\vec{u}}{\partial\vec{x}} + \nabla p - \mu\nabla^2\vec{u} = 0, \nabla \cdot \vec{u} = 0.$$

This system was known as *Oseen's* equations (Onseen [19]). He proposed them as a way of approximately accounting for fluid inertia in problems where there is an ambient free stream $\vec{U}i$. The biharmonic equation can be solved in a neighbourhood of the cylinder (i.e. a circle) by any stream function of the form:

$$\psi = C \sin \theta (r \log r - \frac{1}{2}r + \frac{1}{2r}),$$

but there were no choice of C for which $\psi \sim \sin \theta$ for large r . The analysis of this problem has been covered very well in papers of (Lamb [13] and Lamb, [14]) and its understanding through the use of matched asymptotic expansions(MAE) was one of the triumphs of perturbation theory. However, the MAE approach, despite its immense power and diversity of expanding applications, does not give a clean resolution of the original difficulty in that such methods rely essentially on computing and then matching solutions to the problem defined on two regions: close to the cylinder and far from cylinder.

Mads Kyed came up with an asymptotic structure of a three dimensional Navier-Stokes flow past a rotating body. He considered a body with a connected boundary moving in a Navier-Stokes liquid that fills the whole exterior of the body. Eva Dintelmann, Matthias Geissert and Matthias Heiber [2] considered the Navier-Stokes flow past several moving obstacles with possible time-dependant velocity. It is shown that under suitable assumptions on the data, there exists a unique, local strong solution in the $L^p - L^q$ -setting where $p < q$ and $p, q \in (1, \infty)$. Moreover, it was proved that this strong solution coincides with the unknown mild solution in a very weak sense.

The fundamental problem is to decide whether such smooth, and physically reasonable solutions exist for the Navier-Stokes equations. In two dimensions, the analogues of assertions have been known for a long time (Ladyzhenskaya [12]), and also for the more difficult case of the Euler equations. This gives no hint about the three-dimensional case, since the main difficulties are absent in two dimensions. In three dimensions, it is known that Navier-Stokes equations hold, provided the initial velocity \vec{u}^0 satisfies the smallness condition, for initial data $\vec{u}^0(\vec{x})$ not assumed to be small and it is also known that $f = 0$ hold (also for $\mu = 0$) if the time interval $[0, \infty)$ is replaced by a small time-interval $[0, T)$, with T depending on the initial data. For a given initial velocity $\vec{u}^0(\vec{x})$, the maximum allowable T is called the blowup time.

The above results are covered very well in Bertozzi and Majda [2]. Starting with Leray [15], important progress has been made in understanding weak solutions of Navier-Stokes equations. To arrive at the idea of a weak

solution of a problem, one integrates the equations against a test function, and then integrates by parts (formally) to make the derivatives fall on the test function. The solution of that derivative of the Navier-Stokes equations on test function is called a weak solution of the problem. A long-established idea in analysis is to prove existence and regularity of solutions of a PDE by first constructing a weak formulation of the problem and weak solution. Further we show that any weak solution of the problem is smooth.

This approach has been tried for Navier-Stokes equations with partial success. Leray in [15] showed that the Navier-Stokes equations (when $f = 0$) in three space dimensions always result in weak solutions (p, \vec{u}) with suitable growth properties. The uniqueness of the weak solutions of the Navier-Stokes is not known. For the Euler equation, the uniqueness of the weak solution is false. Scheffer [17] and, later, Schnirelman [22] exhibited weak solutions of the Euler equations on $R^2 \times R$ with compact support in space and time.

Scheffer [20] applied ideas from geometric measure theory to prove a partial regularity theorem for suitable weak solution of the Navier-Stokes equations. Caffarelli-Kohn-Nirenberg [3] improved Scheffer's results, and F-H.lin [16], simplified the proofs of the results in Caffarelli-Kohn-Nirenberg [3]. The partial regularity theorem of Scheffer [20] and [21] concerns a parabolic analogue of the Hausdorff dimension of the singular set of a suitable weak solution of Navier-Stokes. Here, the singular set of a weak solution \vec{u} consists of all points $(\vec{x}^0, t^0 \in R^3 \times R)$ such that \vec{u} is unbounded in every neighborhood of (\vec{x}^0, t^0) .

In this project, we look at the well-posedness of an unsteady state problem of a flow past a rotating obstacle about a fixed axis described by the Navier-Stokes equations. In particular we consider a well known case of well-posedness depending on the size of $\vec{\omega}$. The point of departure with other authors is the use of energy method approach.

Chapter 3

Modelling of Incompressible Navier-Stokes

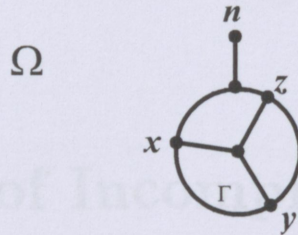


Figure 1.1 A circular cylinder placed in a fluid at rest

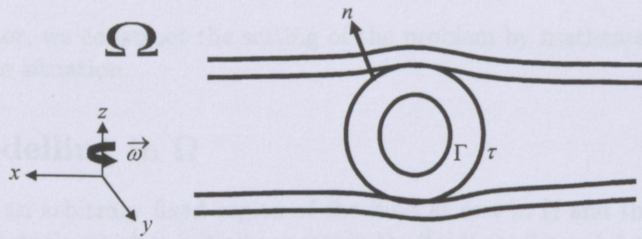


Figure 1.2 A circular cylinder with its boundary Γ rotating with an angular velocity $\vec{\omega}$ in a fluid domain Ω

Since ρ is assumed to be constant, it follows that

$$\nabla \cdot (\rho \vec{u}) = 0. \quad (3.2)$$

Chapter 3

Modelling of Incompressible Navier-Stokes flow

In this chapter, we construct the setting of the problem by mathematically modelling the situation.

3.1 Modelling in Ω

We consider an arbitrary fixed region of the fluid at rest in Ω and the cylindrical obstacle which is initially at rest in the fluid (see figure 1.1 above). The motion of the obstacle is impulsively started and as a result triggering the motion of the surrounding fluid. Fluid particles in-contact with the boundary (Γ) of this cylindrical obstacle in a fluid become embedded in it and then rotate with it, forming a layer τ whose thickness increases with time due to pressure forces in the fluid as a result of rotational motion. Let \vec{n} be the outward unit normal to Γ as shown in figure 1.2 above. If $\rho(\vec{x}, t)$ is the density of the fluid in Ω , and $\vec{u}(\vec{x}, t)$ the velocity of the fluid particle in Ω which, at time t is in position \vec{x} , then the conservation of fluid mass in Ω , may be expressed as:

$$\frac{d}{dt} \int_{CV} \rho d\vec{x} = - \oint_{CS} \rho \vec{u} \cdot \vec{n} ds.$$

For a fixed control volume and by divergence theorem, we have:

$$\begin{aligned} \int_{CV} \rho_t d\vec{x} &= - \int_{CV} \nabla \cdot (\rho \vec{u}) d\vec{x}, \\ \int_{CV} [\rho_t + \nabla \cdot (\rho \vec{u})] d\vec{x} &= 0 \\ \rho_t + \nabla \cdot (\rho \vec{u}) &= 0. \end{aligned}$$

Since ρ is assumed to be constant, it follows that:

$$\nabla \cdot \vec{u}(\vec{x}, t) = 0. \quad (3.0)$$

Eq.(3.0) is referred to as the continuity equation.

Next, we consider the conservation of linear momentum in Ω . This is expressed in the following way:

The rate of change of momentum in the control volume (CV)=the rate of change of momentum over the control surface + vector sum of all surface forces acting on the control volume + all forces acting on both the solid body and fluid in the control volume.

The next task is to choose appropriate flux terms and the unsteady control volume term. Since we are dealing with a fixed control volume, we choose the momentum flux \vec{M} so that the rate of change of linear momentum over the control surface, CS, with respect to the outward unit normal \vec{n} is given as:

$$\int_{CS} \vec{M} \cdot \vec{n} ds = - \int_{CV} \nabla \cdot \vec{M} d\vec{x}.$$

Let \vec{T} denote the stress tensor on Γ . The (internal) force per unit area at $\vec{x} \in CS$, may be expressed in the form $\vec{T}(\vec{x}) \cdot \vec{n}(\vec{x})$. Thus, the total force on CS is:

$$\int_{CS} \vec{T} \cdot \vec{n} ds = \int_{CV} \nabla \cdot \vec{T} d\vec{x},$$

by the divergence theorem.

We then write the conservation law of linear momentum of the fluid as:

$$\frac{d}{dt} \int_{CV} \rho \vec{u} d\vec{x} = - \int_{CV} \nabla \cdot \vec{M} d\vec{x} + \int_{CV} \nabla \cdot \vec{T} d\vec{x} + \int_{CV} \rho \vec{f} d\vec{x}.$$

Since Ω is arbitrary,

$$\partial_t(\rho \vec{u}) + \nabla \cdot \vec{M} = \nabla \cdot \vec{T} + [(\vec{\omega} \times \vec{x}) \cdot \nabla] \vec{u} + \rho \vec{f}, \quad (3.1)$$

where,

$$[(\vec{\omega} \times \vec{x}) \cdot \nabla] \vec{u},$$

is a drift operator of a rotating body due to conservation of angular momentum. Now choose \vec{M} as follows:

$$\vec{M} = \rho \vec{u} \times \vec{u},$$

where the symbol (\times) denotes the vector product. Having made this choice, we have:

$$\nabla \cdot \vec{M} = [\nabla \cdot (\rho \vec{u} \times \vec{u})] \quad (3.2)$$

$$\nabla \cdot (\rho \vec{u} \times \vec{u}) = \sum_{i=1, j=1}^3 \partial_i (\rho \vec{u}_i \cdot \vec{u}_j) \quad (3.3)$$

$$= \sum_{i=1}^3 [\partial_i (\rho \vec{u}_i)] \cdot \vec{u}_j + \sum_{j=1}^3 \rho \vec{u}_i \partial_i \cdot \vec{u}_j \quad (3.4)$$

$$= [\nabla \cdot (\rho \vec{u})] \vec{u} + \rho (\vec{u} \cdot \nabla) \vec{u}. \quad (3.5)$$

Hence,

$$\nabla \cdot \vec{M} = [\nabla \cdot (\rho \vec{u})] \vec{u} + \rho (\vec{u} \cdot \nabla) \vec{u}. \quad (3.6)$$

But then,

$$\partial_t (\rho \vec{u}) = \rho_t \vec{u} + \rho \vec{u}_t = (-\nabla \cdot \rho \vec{u}) \vec{u} + \rho \vec{u}_t, \quad (3.7)$$

by (3.0) we write (3.1) using equations (3.6) and (3.7) above to obtain the expression:

$$\rho (\vec{u} \cdot \nabla) \vec{u} = \nabla \cdot \vec{T} + (\vec{\omega} \times \vec{x}) \cdot \nabla \vec{u} + \rho \vec{f},$$

which gives

$$\rho \vec{u}_t + \rho (\vec{u} \cdot \nabla) \vec{u} = \nabla \cdot \vec{T} + (\vec{\omega} \times \vec{x}) \cdot \nabla \vec{u} + \rho \vec{f}; \quad (3.8)$$

which is the mathematical statement of conservation of linear momentum and angular momentum combined with the conservation of mass.

3.2 Boundary conditions

In this section we discuss the boundary conditions as part of the governing equations for the motion of a rotating obstacle in an incompressible flow. We denote it by Γ . This boundary of an obstacle Γ is assumed to be impermeable, i.e. it does not allow flow of the fluid across it.

The geometry of the problem under investigation is shown in Figure 1.2 above. The outward normal vector to Γ is given by \vec{n} . The fluid velocity vector at the surface of an obstacle, is given by $\gamma_0 \vec{u}(\vec{x}, t)$. This velocity is equal to zero at the stagnation point when the fluid is in contact with an obstacle. Our model assumes that the boundary of the obstacle is

impermeable and the fluid particles adjacent to it become embedded to the surface of the obstacle and rotate with it.

We have already derived the equations of motion in the previous section. We have for incompressible fluid:

$$\int D_t \vec{u}(\vec{x}, t) = \nabla \cdot \vec{T}(p, \vec{u}); \vec{x}, \vec{u} \in \Omega,$$

$$\nabla \cdot \vec{u}(\vec{x}, t) = 0, \quad (3.9)$$

where,

$$D_t := \partial_t + \vec{u} \cdot \nabla,$$

and for an incompressible Navier-Stokes fluid, the stress tensor is chosen as

$$\vec{T}(p, \vec{u}) = -pI + 2\mu \mathbf{D}\vec{u},$$

where $\mathbf{D}(\vec{u}) = \frac{1}{2}[\nabla \vec{u} + \nabla \vec{u}^T]$, is called the rate of deformation tensor. Thus our conditions are:

$$\vec{u}(\vec{x}, 0) = 0 \text{ in } \Omega,$$

$$\vec{u}_\Gamma = \vec{\omega} \times \vec{x} \text{ on } \Gamma.$$

3.3 Problem Formulation

The purpose of this work is to establish well-posedness of the 3-dimensional fluid flow past a rotating obstacle about a fixed axis problem for small Re (i.e. $Re \ll 1$) by using the energy method. We consider the case of rotation only, i.e. without translation. Having modelled the motion of the fluid in Ω and discussed the boundary condition on Γ , we are now ready to formulate our problem. The full system of equations modelling the motion of the fluid past a rotating obstacle is given below as:

$$\rho \partial_t \vec{u} + \rho(\nabla \cdot \vec{u})\vec{u} = -\nabla p + \mu \Delta \vec{u} + [(\vec{\omega} \times \vec{x}) \cdot \nabla] \vec{u} + \rho \vec{f} \quad (3.10)$$

in Ω

$$\vec{u}_\Gamma = \vec{\omega} \times \vec{x} \text{ on } \Gamma; \quad (3.11)$$

$$\gamma_0 \vec{u}(\vec{x}, t) = 0 \text{ on } \Gamma \text{ (at the stagnation point);} \quad (3.12)$$

$$\nabla \cdot \vec{u}(\vec{x}, t) = 0 \text{ in } \Omega; \quad (3.13)$$

$$\vec{u}^0(\vec{x}, 0) = \vec{u}^0(\vec{x}) = 0 \text{ in } \Omega \quad (3.14)$$

The unknowns in the above system are $\vec{u}(\vec{x}, t)$ (the Eulerian velocity of the fluid motion), $p(\vec{x}, t)$ (the pressure of the fluid). $\vec{f}(\vec{x}, t)$ is the force acting on the fluid and $[(\vec{\omega} \times \vec{x}) \cdot \nabla] \vec{u}$ is the drift operator of the rotating body. For $x, y \in R^n$, the notation $\langle x, y \rangle$ stands for inner product of x and y and $|x|$ stands for its corresponding norm. We denote by Γ the boundary of the rigid body. As far as we know, a local in time existence result of mild solutions (in three spatial dimensions) has been proved by Hishida [8] for initial data possessing the same regularity as in the paper of Fujita and Kato [13]. The above mentioned paper uses a non-trivial generalization of the semigroup method of Fujita and Kato [13]. Moreover, very recently, a local in time existence and uniqueness result of strong solutions (in three spatial dimensions) has been proved by Galdi and Silvestre [6]. In that paper, the authors show that if the initial velocity \vec{u}_0 , in a suitable norm, and the magnitude of $\vec{\omega}$ do not exceed a certain constant depending only on the viscosity and on the regularity of the solution it is global in time. Both studies mentioned above deal with the problem by writing the equations of the system in a frame attached to the obstacle. The aim of this project is to establish well posedness of a weakly formulated problem in 3-dimensions by using the so called energy method.

3.3.1 Energy method

The energy method describes the flow motion in terms of the energy equations associated with that motion. In our case the domain of the flow is unbounded, and the velocity of fluid flow differ from one point to the other. However, we can still calculate the energy of small control volumes of the domain Ω , add them together and pass to a limit in which the control volume is squeezed to zero. This results in the integral:

$$e(\vec{u}) = \frac{1}{2} \int_{\Omega} |\vec{u}|^2 d\vec{x}.$$

Assume that initial data vanishes outside a large radius of the flow, so that the energy integral is convergent due to finite velocity of the flow. Then we check to see whether the kinetic energy is conserved in time. Clearly, since $e = \frac{1}{2} \|\vec{u}\|_{L^2(\Omega)}^2$ if $\vec{u} \in L^2(\Omega)$ the flow has finite energy.

We use this method, in order to show the local existence and uniqueness of a weak solution inspired by the approach in Reference [6],[7] and [8]. In Reference [6] the author proved the existence and uniqueness of a strong solution in two spatial dimensions. In Reference [8] the author proved the local in time existence of a mild solution in three spatial dimensions. The key step to our approach is to change the whole Navier-stokes equation into an energy equation and show that the total energy is conserved .

If the kinetic energy is conserved, then the total (kinetic) energy in the

domain does not grow with time (provided the energy fluxes at the boundaries are null/bounded). This ensures that the velocity at every point in the domain is bounded.

Now to derive the energy form for the problem, we take the scalar product of $u(x, t) \in \Omega$ with equation (3.10) and obtain:

$$\rho \partial_t \vec{u}, \vec{u} + \rho((\nabla \cdot \vec{u}), \vec{u}) = -\nabla p, \vec{u} + \mu \Delta \vec{u}, \vec{u} + [(\vec{\omega} \times \vec{x}) \cdot \nabla] \vec{u}, \vec{u} + \rho \vec{f}, \vec{u}$$

Now dealing with each term of the above equation after it has been multiplied by $\vec{u}(\vec{x}, t)$, gives transformed terms of it which we use to obtain an energy equation equivalent to it. To achieve that partially we consider and prove the following Lemma in order to transform the first L.H.S term of it:

3.3.2 Lemma

Let H be a real Hilbert Space and $A : H \rightarrow H$ a linear (not necessarily bounded or closed) operator. For $u \in D(A)$, let the bilinear form r be defined by $r(u, v) := (u, Av)$. Let us suppose that r is symmetrical for $u, v \in D(A)$? If $t \rightarrow v(t) \in D(A)$ is differentiable in the norm and $t \rightarrow Av(t)$ is weakly continuous, then the quadratic form $\hat{r}(v(t))$ is differentiable and

$$\frac{d\hat{r}(v(t))}{dt} = 2r(v'(t), v(t))$$

Proof:

$$\begin{aligned} \frac{d\hat{r}(v(t))}{dt} &= \lim_{h \rightarrow 0} h^{-1} [(v(t+h), Av(t+h)) - (v(t), Av(t))] \\ &= \lim_{h \rightarrow 0} h^{-1} [(v(t+h), Av(t+h)) - (v(t), Av(t+h)) \\ &\quad + (v(t), Av(t+h)) - (v(t), Av(t))] \\ &= (\lim_{h \rightarrow 0} h^{-1} [v(t+h) - v(t)], \lim_{h \rightarrow 0} Av(t+h)) \\ &\quad + \lim_{h \rightarrow 0} (v(t), h^{-1} [Av(t+h) - Av(t)]) \\ &= \left(\frac{dv}{dt}, Av(t) \right) + \left(\lim_{h \rightarrow 0} h^{-1} [v(t+h) - v(t)], Av(t) \right) \\ &= 2 \frac{d}{dt} v Av(t) \end{aligned}$$

since $\hat{r}(u, v)$ is symmetrical. The first term of the equation on the left hand side becomes:

$$\rho(\partial_t \vec{u}(x, t), \vec{u}(x, t))_{L^2(\Omega)} = \frac{1}{2} \rho \frac{d}{dt} \|\vec{u}(x, t)\|_{L^2(\Omega)}^2$$

The second term of left hand side of it becomes:

$$\begin{aligned}
\rho((\vec{u} \cdot \nabla)\vec{u}, \vec{u})_{L^2(\Omega)} &= \rho \int_{\Omega} [(\vec{u} \cdot \nabla)\vec{u}, \vec{u}] \\
&= \rho \sum_{m=1}^3 \sum_{k=1}^3 \frac{1}{2} \int_{\Omega} \vec{u}_k \frac{\partial(\vec{u}_m \vec{u}_i)}{\partial x_k} dx_m \\
&= \frac{\rho}{2} \left[\int_{\Gamma} (|\gamma_0 \vec{u} \cdot \vec{n}| |\gamma_0 \vec{u}|^2) ds - \int_{\Omega} (|\vec{u}|^2 \nabla \cdot \vec{u}) dx \right] = 0
\end{aligned}$$

The first term on the right hand side of it is given by:

$$\begin{aligned}
-\nabla p, \vec{u} &= \int_{\Omega} \nabla p \cdot \vec{u} dx \\
&= \int_{\Gamma} p \gamma_0 \vec{u} \cdot \vec{n} ds + \int_{\Omega} p (\nabla \cdot \vec{u}) dx = 0
\end{aligned}$$

The second term on the right hand side of it takes the form:

$$\mu(\Delta \vec{u}, \vec{u})_{L^2(\Omega)} = \mu \int_{\Omega} (\vec{u} \Delta \vec{u}) dx = \mu \int_{\Gamma} \gamma_0 \vec{u} \cdot \vec{n} (\gamma_0 \vec{u}) ds - \frac{\mu}{2} \int_{\Omega} (\nabla \vec{u} \cdot \nabla \vec{u}) dx$$

by equation (3.12) above:

$$\mu \int_{\Gamma} \gamma_0 \vec{u} \cdot \vec{n} (\gamma_0 \vec{u}) ds = 0$$

Hence:

$$\mu(\Delta \vec{u}, \vec{u})_{L^2(\Omega)} = -\frac{\mu}{2} \int_{\Omega} (\nabla \vec{u} \cdot \nabla \vec{u}) dx = -\frac{\mu}{2} \|\nabla \vec{u}\|_{L^2(\Omega)}^2$$

The third term on the right hand side of it takes the form:

$$\begin{aligned}
((\vec{\omega} \times \vec{x}) \cdot \nabla \vec{u}), \vec{u} &= (\gamma_0 \vec{u} \cdot \nabla \vec{u}), \vec{u} \\
&= \left[\int_{\Gamma} (\gamma_0 \vec{u} \cdot \vec{n} \cdot \gamma_0 \vec{u}) ds - \int_{\Omega} \gamma_0 \vec{u} (\nabla \cdot \vec{u}) dx \right], \vec{u} = 0
\end{aligned}$$

Putting together the transformed terms of the equation, we obtain the following equation:

$$\frac{\rho}{2} \frac{d}{dt} \|\vec{u}\|_{L^2(\Omega)}^2 + \frac{\mu}{2} \|\nabla \vec{u}\|_{L^2(\Omega)}^2 = \rho(f, \vec{u})_{L^2(\Omega)}$$

which is the energy form of the Navier-Stokes equation describing the flow in question. Thus,

$$\Theta := \{ \vec{u}(\vec{x}, t) \in H^2(\Omega) \times [0, T) : \nabla \cdot \vec{u}(\vec{x}, t) = 0, \vec{u}_{\Gamma}(\vec{x}, t) = \vec{\omega} \times \vec{x}, \vec{x} \in \Omega \},$$

which has a compact support in Ω becomes our test function. Starting with Leray [15], important progress has been made in understanding weak solutions of Navier-stokes equations. To arrive at the idea of a weak solution of a problem, one integrates the equations by using a test function. The methods of integration by parts is applied formally with derivatives falling on the test function. The solution of the resulting problem on test function is called a weak solution of the problem. In the next chapter, we solve the the existence and uniqueness of the equation

$$\frac{\rho}{2} \frac{d}{dt} \|\vec{u}\|_{L^2(\Omega)}^2 + \frac{\mu}{2} \|\nabla \vec{u}\|_{L^2(\Omega)}^2 = \rho(f, \vec{u})_{L^2(\Omega)},$$

by applying Riesz's representation Theorem. We solve it by finding the boundedness of the velocity of the fluid flow.

Chapter 4

Existence and Uniqueness of the Solution of the problem

In the previous chapter we derived the energy form of the equations governing the problem. In this chapter we prove the existence and uniqueness of the solution to the problem by studying the energy form of the equations describing the problem:

$$\frac{\rho}{2} \frac{d}{dt} \|\bar{u}\|_{L^2(\Omega)}^2 + \frac{\mu}{2} \|\nabla \bar{u}\|_{L^2(\Omega)}^2 = \rho(f, \bar{u})_{L^2(\Omega)}, \quad (4.1)$$

obtained in the previous section. let us re-write equation (4.1) in the form:

$$E'(t) + \mu \|\nabla \bar{u}\|_{L^2(\Omega)}^2 = \rho(f, \bar{u})_{L^2(\Omega)}, \quad (4.2)$$

where,

$$E'(t) = \frac{\rho}{2} \frac{d}{dt} \|\bar{u}\|_{L^2(\Omega)}^2.$$

In order to simplify $\|\nabla \bar{u}\|_{L^2(\Omega)}^2$ of (4.2), we use the following corollary:

An important question in the theory of Sobolev spaces is whether one can estimate the norm of a function in terms of the norm of its first derivatives. The following result provides an elementary estimate in this direction. It is valid for domains Ω such that:

$$\Omega \subseteq \{x = x_1, x_2, \dots, x_n\}$$

Corollary: For $c_\rho > 0$ and since the control volume in Ω is fixed, we can deduce the poincare inequality:

$$\|\nabla \bar{u}(t)\|_{L^2(\Omega)}^2 \geq c_\rho \|\bar{u}(t)\|_{L^2(\Omega)}^2 \quad (\text{see pp.248-249 of [9]}) \quad (4.3)$$

In view of (4.3) above, we write (4.2) as follows:

$$E'(t) + \mu c_\rho \|\bar{u}(t)\|_{L^2(\Omega)}^2 \leq 2\rho(f, \bar{u})_{L^2(\Omega)}. \quad (4.4)$$

Re-writing the inequality (4.4) above in terms of the kinetic energy for the flow, we obtain the following first order linear differential inequality:

$$E'(t) + \frac{\mu c_\rho}{\rho} E(t) \leq 2\rho(f, \bar{u})_{L^2(\Omega)}. \quad (4.5)$$

The solution to (4.5) is given by,

$$E(t) \leq \exp\left(\frac{-\mu c_\rho t}{\rho}\right) \left[\int_0^t \exp\left(\frac{\mu c_\rho}{\rho}\right) (f(\bar{x}, t), \bar{u}(\bar{x}, t))_{L^2(\Omega)} + C_E \exp\left(\frac{-\mu c_\rho t}{\rho}\right) \right] \quad (4.6)$$

$$t \in [0, T]$$

Remarks: The inequality (4.6) implies that $E(0) \leq C_E$, $t = 0$
This, in turn, implies that,

$$\|\bar{u}^0(x)\| \leq \frac{C_E}{\rho}. \quad (4.7)$$

Further, we have ,

$$\|D(\bar{u})\|_{L^2(\Omega)}^2 \geq \frac{1}{4} \|\nabla \bar{u}\|_{L^2(\Omega)}^2 \geq \frac{1}{4} c_\rho \|\bar{u}\|_{L^2(\Omega)}^2, \text{ see equation (12) of Hlomu\u010dka [10]} \quad (4.8)$$

By (11) on page 9 of Hlomu\u010dka [9],

$$E'(t) \leq -\|\bar{u}\|_{L^2(\Omega)} \beta(t),$$

where $\beta(t) > 0$; for all $t \leq 0$; is given by,

$$\beta(t) := 1 - C_2(2E(t))^{\frac{1}{2}} - C_2^2 E(t); C_2 > 0;$$

$E(t)$:=energy problem at time t .

Using the energy problem equation above, we get:

$$E'(t) \leq -\mu \|D(\bar{u})\| \beta(t) \quad (4.9)$$

But then:

$$\frac{d\beta(t)}{dE(t)} = -C_2^2 - \frac{1}{2} C_2 \sqrt{2E}^{-\frac{1}{2}}(t),$$

Thus for the maximum $\beta(t)$, we put $\frac{d\beta(t)}{dE(t)} = 0$ and obtain,

Max $\beta(t) = \frac{3}{2}$ at $E(t) = \frac{1}{2C_2^2}$ (See the bottom of page 9 in Hlomu\u010dka [9]).

Then

$$E'(t) \geq \frac{-3}{8} \mu c_\rho \|\bar{u}\|_{L^2(\Omega)}^2. \quad (4.10)$$

The above inequality, implies that,

$$\frac{-3}{8} \mu c_\rho \|\vec{u}\|_{L^2(\Omega)}^2 + \mu \|\nabla \vec{u}\|_{L^2(\Omega)}^2 \leq 2\rho(f, \vec{u})_{L^2(\Omega)}. \quad (4.11)$$

At the end of chapter 3, we had:

$$\frac{\mu}{2} \|\nabla \vec{u}\|_{L^2(\Omega)}^2 = -\mu(\Delta \vec{u}, \vec{u})_{L^2(\Omega)}, \quad (4.12)$$

which implies that:

$$\mu \|\nabla \vec{u}\|_{L^2(\Omega)}^2 = -2\mu(\Delta \vec{u}, \vec{u})_{L^2(\Omega)}. \quad (4.13)$$

Using (4.13), we re-write (4.11) and obtain,

$$\frac{-3}{8} \mu c_\rho \|\vec{u}\|_{L^2(\Omega)}^2 - 2\mu(\Delta \vec{u}, \vec{u})_{L^2(\Omega)} \leq 2\rho(f, \vec{u})_{L^2(\Omega)}. \quad (4.14)$$

4.1 Riesz's representation for the problem

We re-write (4.14) as follows:

$$\frac{-3}{8} \mu c_\rho (\vec{u}, \vec{u})_{L^2(\Omega)} - 2\mu(\Delta \vec{u}, \vec{u})_{L^2(\Omega)} \leq 2\rho(f, \vec{u})_{L^2(\Omega)}. \quad (4.15)$$

This implies that:

$$\left(\left(\frac{-3}{8} \mu c_\rho I - 2\mu\Delta \right) \vec{u}, \vec{u} \right)_{L^2(\Omega)} \leq 2\rho(f, \vec{u})_{L^2(\Omega)}. \quad (4.16)$$

Remarks:

- (a) Until now, through the application of Poincaré inequality theorem, our aim has been to establish the boundedness of the right hand side of (4.16).
- (b) It is not hard to show that the left hand side of (4.16) is a bounded sesquilinear form.
- (c) Thus, by the Riesz's representation theorem [15], there exists a bounded linear operator A such that,

$$\left(\left(\frac{-3}{8} \mu c_\rho I - 2\mu\Delta \right) \vec{u}, \vec{v} \right)_{L^2(\Omega)} = (A\vec{u}, \vec{v})_{L^2(\Omega)}, \quad (4.17)$$

from which we conclude that:

$$\frac{-3}{8} \mu c_\rho I - 2\mu\Delta = A. \quad (4.18)$$

Remarks: To prove the uniqueness of (4.18), first we must show that:

- (a) it is self-adjoint;
- (b) inverse exists and is bounded;
- (c) it is compact;
- (d) inverse is also compact; and
- (e) it is uniformly bounded.

4.2 Characterisation of the operator $\frac{-3}{8}\mu c_\rho I - 2\mu\Delta$

Proposition 4.2.1: The operator $\frac{-3}{8}\mu c_\rho I - 2\mu\Delta$ is self-adjoint and positive on Θ . Since $\frac{-3}{8}\mu c_\rho - 2\mu$ is constant, then our operator is Δ .

Proof: Let $u, v \in \Theta$, Δ^* be adjoint of Δ and for $u = v$

$$\begin{aligned} (\Delta \vec{u}, \vec{v})_{L^2(\Omega)} &= \int_{\Gamma} \gamma_0 \vec{v} \cdot \vec{n} \nabla_s(\gamma_0 \vec{u}) ds - \frac{1}{2} \int_{\Omega} (\nabla \vec{u} \cdot \nabla \vec{v}) dx, \\ (\vec{u}, \Delta^* \vec{v})_{L^2(\Omega)} &= \int_{\Gamma} \gamma_0 \vec{u} \cdot \vec{n} \nabla_s^*(\gamma_0 \vec{v}) ds - \frac{1}{2} \int_{\Omega} (\nabla^* \vec{v} \cdot \nabla \vec{u}) dx, \end{aligned}$$

Thus,

$$(\Delta \vec{u}, \vec{v})_{L^2(\Omega)} = (\vec{u}, \Delta^* \vec{v})_{L^2(\Omega)},$$

implies that,

$$\int_{\Gamma} \gamma_0 \vec{v} \cdot \vec{n} \nabla_s(\gamma_0 \vec{u}) ds - \frac{1}{2} \int_{\Omega} (\nabla \vec{u} \cdot \nabla \vec{v}) dx = \int_{\Gamma} \gamma_0 \vec{u} \cdot \vec{n} \nabla_s^*(\gamma_0 \vec{v}) ds - \frac{1}{2} \int_{\Omega} (\nabla^* \vec{v} \cdot \nabla \vec{u}) dx,$$

and since,

$$\int_{\Gamma} \gamma_0 \vec{v} \cdot \vec{n} \nabla_s(\gamma_0 \vec{u}) ds = \int_{\Gamma} \gamma_0 \vec{u} \cdot \vec{n} \nabla_s^*(\gamma_0 \vec{v}) ds = 0,$$

due to $\gamma_0 \vec{u} \cdot \vec{n} = 0$. Thus,

$$-\frac{1}{2} \int_{\Omega} (\nabla \vec{u} \cdot \nabla \vec{v}) dx = -\frac{1}{2} \int_{\Omega} (\nabla^* \vec{v} \cdot \nabla \vec{u}) dx.$$

The latter equality implies that:

$$\nabla \vec{v} = \nabla^* \vec{v}.$$

This proves the self-adjointness of the operator. Further, we consider:

$$\begin{aligned}
\left(\left(\frac{-3}{8} \mu c_\rho I - 2\mu \Delta \right) \vec{u}, \vec{u} \right)_{L^2(\Omega)} &= \frac{-3}{8} \mu c_\rho (\vec{u}, \vec{u})_{L^2(\Omega)} - 2\mu (\Delta \vec{u}, \vec{u})_{L^2(\Omega)} \\
&= \frac{-3}{8} \mu c_\rho \|\vec{u}\|_{L^2(\Omega)}^2 - 2\mu \\
&\quad \left[\int_{\Gamma} \gamma_0 \vec{u} \cdot \vec{n} \nabla_s (\gamma_0 \vec{u}) ds - \frac{1}{2} \int_{\Omega} (\nabla \vec{u} \cdot \nabla \vec{u}) dx \right]
\end{aligned}$$

Thus,

$$\begin{aligned}
\left(\left(\frac{-3}{8} \mu c_\rho I - 2\mu \Delta \right) \vec{u}, \vec{u} \right)_{L^2(\Omega)} &= \frac{3}{8} \mu (-) c_\rho \|\vec{u}\|_{L^2(\Omega)}^2 + \mu \int_{\Omega} (\nabla \vec{u} \cdot \nabla \vec{u}) dx - \\
&\quad 2\mu \left[\int_{\Gamma} \gamma_0 \vec{u} \cdot \vec{n} \nabla_s (\gamma_0 \vec{u}) ds \right] \\
&= \mu \|\nabla \vec{u}\|_{L^2(\Omega)}^2 - \frac{3}{8} \mu c_\rho \|\vec{u}\|_{L^2(\Omega)}^2 - \\
&\quad 2\mu \left[\int_{\Gamma} \gamma_0 \vec{u} \cdot \vec{n} \nabla_s (\gamma_0 \vec{u}) ds \right].
\end{aligned}$$

At the contact between the fluid flow and rotating obstacle, we have

$$\gamma_0 \vec{u} \cdot \vec{n} = 0.$$

Thus,

$$\begin{aligned}
\left(\left(\frac{-3}{8} \mu c_\rho I - 2\mu \Delta \right) \vec{u}, \vec{u} \right)_{L^2(\Omega)} &= \mu \|\nabla \vec{u}\|_{L^2(\Omega)}^2 - \frac{3}{8} \mu c_\rho \|\vec{u}\|_{L^2(\Omega)}^2 \\
&\geq \mu c_\rho \|\vec{u}\|_{L^2(\Omega)}^2 - \frac{3}{8} \mu c_\rho \|\vec{u}\|_{L^2(\Omega)}^2.
\end{aligned}$$

On applying the poincare inequality, we have:

$$\begin{aligned}
\left(\left(\frac{-3}{8} \mu c_\rho I - 2\mu \Delta \right) \vec{u}, \vec{u} \right)_{L^2(\Omega)} &\geq \frac{5}{8} \mu c_\rho \|\vec{u}\|_{L^2(\Omega)}^2 \\
&\geq 0.
\end{aligned}$$

Hence this proves that the operator is positive on Θ .

Proposition 4.2.2: $\left(\frac{-3}{8} \mu c_\rho I - 2\mu \Delta \right)$ is invertible and $\left(\frac{-3}{8} \mu c_\rho I - 2\mu \Delta \right)^{-1}$ is a bounded linear operator on Θ .

Proof: Let $\vec{u} \in \ker(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)\vec{u}$. Then,

$$(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)\vec{u} = 0,$$

this implies that

$$\frac{-3}{8}\mu c_\rho \vec{u} - 2\mu\Delta\vec{u} \Leftrightarrow \frac{-3}{8}\mu c_\rho \vec{u} = 2\mu\Delta\vec{u} = 0 \Leftrightarrow \vec{u} = 0.$$

Then,

$$(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta) = 0,$$

and hence the inverse:

$$(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}$$

exists.

To prove that the inverse operator is bounded, we first prove that it is linear:

$$1. (\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}(\vec{u}_1 + \vec{u}_2) = (\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}\vec{u}_1 + (\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}\vec{u}_2$$

$$2. (\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}(\lambda\vec{u}) = \lambda(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}\vec{u}$$

Now to prove (1): we put

$$(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}(\vec{u}_1 + \vec{u}_2) = \vec{w}, \vec{w} \in \Theta,$$

then

$$(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)\vec{w} = \vec{u}_1 + \vec{u}_2$$

again

$$\begin{aligned} (\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}\vec{u}_1 &= \vec{w}_1, \vec{w}_1 \in \Theta, \\ (\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}\vec{u}_2 &= \vec{w}_2, \vec{w}_2 \in \Theta \end{aligned}$$

then

$$(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)\vec{w}_1 = \vec{u}_1,$$

and,

$$\left(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right)\vec{w}_2 = \vec{u}_2,$$

therefore

$$\vec{w}_1 + \vec{w}_2 = \left(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right)^{-1}\vec{u}_1 + \left(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right)^{-1}\vec{u}_2,$$

and

$$\vec{u}_1 + \vec{u}_2 = \left(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right)\vec{w}_1 + \left(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right)\vec{w}_2,$$

since

$$\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta,$$

is linear, then

$$\vec{w}_1 + \vec{w}_2 = \left(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right)^{-1}(\vec{u}_1 + \vec{u}_2),$$

Substituting it in place of,

$$\vec{w}_1 + \vec{w}_2$$

we have the results. To prove (2): put

$$\left(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right)^{-1}\vec{u} = \vec{w}, \vec{w} \in \Theta,$$

then

$$\vec{u} = \left(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right)\vec{w}$$

Multiplying both sides by a constant λ , we have

$$\lambda\vec{u} = \lambda\left(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right)\vec{w}$$

$$\lambda\vec{u} = \left(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right)\lambda\vec{w}$$

$$\lambda\vec{u} = \left(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right)^{-1}(\lambda\vec{u})$$

Replacing, \vec{w} , by,

$$\left(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right)^{-1}\vec{u},$$

the results follow. In view of the corollary on page 96 of [17],

$$\left(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right)^{-1}\vec{u},$$

is bounded on Θ .

Proposition 4.2.3: $\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta$ is a compact operator on Θ .

Proof: Since $\Omega \subset R^3$ and Θ is finite dimensional space. By the corollary on page 407 of [17], $\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta$ is compact on Θ . By the Reisz Representation Theorem [17], we conclude that A is also compact on Θ .

Proposition 4.2.4: The operator $\left(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right)^{-1}A$ is compact on Θ .

Proof: By the Reisz Representation Theorem of Kreyszig [17] and Proposition 4.2.3, A is a compact on Θ . By proposition 4.2.2, $\left(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right)^{-1} : \Theta \rightarrow \Theta$ is a bounded linear operator. By theorem 8.3-2 in [17], then $\left(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right)^{-1}A$ is also compact and the results follow.

4.3 Uniqueness of solution of the problem

Next, using (4.18), and in keeping with the requirements by the leray-Schauder fixed-point theorem of Zeidler [26], we construct the following form for the problem:

$$\left(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right)^{-1}\lambda\vec{u}, \text{ where } \lambda \in (0, 1).$$

Our aim is to show that the equation above has a unique leray-Schauder fixed-point [26], which is the solution to the problem. First, we state and prove the lemma:

4.3.1 Lemma

The solution to $\left(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right)^{-1}\lambda\vec{u}$ is uniformly bounded in Θ .

Proof: $\vec{u} = \lambda\left(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right)^{-1}A\vec{u}$ where $\lambda \in (0, 1)$, implies that,

$$\left(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right)\vec{u} = \lambda A\vec{u},$$

which, in turn, implies that:

$$\left\|\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta\right\|_{W^2(\Omega)}\|\vec{u}\|_{W^2(\Omega)} \leq |\lambda|\|A\vec{u}\|_{W^2(\Omega)} \leq \|A\vec{u}\|_{W^2(\Omega)} \text{ since } |\lambda| < 1$$

By the Riesz Representation theorem, A is also bounded, and, hence, there

exists $\eta > 0$, such that $\|A\bar{u}\|_{W^2(\Omega)} \leq \eta$.

By $\| \frac{-3}{8} \mu c_\rho I - 2\mu\Delta \|_{W^2(\Omega)} \| \bar{u} \|_{W^2(\Omega)} \leq |\lambda| \| A\bar{u} \|_{W^2(\Omega)} \leq \| A\bar{u} \|_{W^2(\Omega)}$ since $|\lambda| < 1$.

This implies that,

$$\| \bar{u} \|_{W^2(\Omega)} \leq \frac{\eta}{\| \frac{-3}{8} \mu c_\rho I - 2\mu\Delta \|_{W^2(\Omega)}}, \text{ and hence the velocity is bounded.}$$

Main Theorem: There exists a unique fixed-point for $(\frac{-3}{8} \mu c_\rho I - 2\mu\Delta)^{-1} \lambda \bar{u}$, which is a solution to (3.10).

Proof: By lemma 4.3.1 and proposition 4.2.4, according to the Leray-Schauder fixed-point theorem [26], the equation has a uniformly bounded solution. To prove uniqueness:

Suppose $(\frac{-3}{8} \mu c_\rho I - 2\mu\Delta)^{-1} \lambda \bar{u}$ has two solutions \bar{u} and \bar{v} , then,

$$\begin{aligned} \| \bar{u} - \bar{v} \|_{W^2(\Omega)} &= \lambda \| (\frac{-3}{8} \mu c_\rho I - 2\mu\Delta)^{-1} A(\bar{u} - \bar{v}) \|_{W^2(\Omega)} \\ &\leq |\lambda| \| (\frac{-3}{8} \mu c_\rho I - 2\mu\Delta)^{-1} A \|_{W^2(\Omega)} \| \bar{u} - \bar{v} \|_{W^2(\Omega)} \end{aligned}$$

due to boundedness

$$\begin{aligned} &< |\lambda| \| (\frac{-3}{8} \mu c_\rho I - 2\mu\Delta)^{-1} A \|_{W^2(\Omega)} \| \bar{u} - \bar{v} \|_{W^2(\Omega)}; 0 < |\lambda| < 1 \\ &= I \| \bar{u} - \bar{v} \|_{W^2(\Omega)}; \end{aligned}$$

which is impossible. Hence

$$\bar{u} = \bar{v}$$

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Chapter 5

Discussions and Conclusions

The aim of the study was to implement the energy method to establish well posedness of the problem describing the motion of a rotating obstacle induced fluid flow. The application of the method has confirmed the earlier results by other authors like Giovanni P.Galdi [6], Toshiaki Hishida [8] and others who used different approaches to establish existence and uniqueness of the solution to the problem being considered here. The method was implemented successfully to achieve the following:

- (a) Existence of a smooth velocity and pressure for the flow induced by an impulsively started motion of a rotating solid circular cylinder.
- (b) confirmation of the boundedness of the velocity of the flow for the motion initially started with small angular velocity.

The results of propositions (4.2.1)-(4.2.4) confirm the existence and uniqueness of the weak solution of the problem. The pressure has been eliminated in the energy statement of the problem. But, then, if the velocity solution existed and is unique, the same would apply to pressure. Mathematically, once the velocity has been proven to exist and unique, we could use the Navier-Stokes equation to find the corresponding pressure.

In conclusion, we say that the velocity of the fluid flow decays uniformly and is bounded. This implies that the rotating obstacle initiated flow velocity dies away as time increases indefinitely. What will make the velocity of a rotating obstacle to vanish is that, initially the body is given small angular velocity to start the motion. Then as it rotates, the fluid layer is formed around it and as a result the diameter of the body is increased. However, the resulting heavier body need to be driven by the same small angular velocity. This leads to the slowing down of the motion of the obstacle until it eventually comes to a stop.

As part of our contribution to knowledge, we have demonstrated the

feasibility of the application of the energy method in establishing well posedness of the problem of the impulsively started motion of a rotating obstacle induced fluid flow. This work can be extended to the investigation of the global existence and uniqueness of the solution for the same problem set up which might also include body translation.

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