



University of Venda

**DEPARTMENT OF MATHEMATICS
AND
APPLIED MATHEMATICS**

**Numerical Simulations of the Stokes Flow by the
Iterations of Boundary Conditions and Finite Difference
Methods**

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Abstract

In this study the iteration of boundary conditions method (Chizhonkov and Kargin, 2006) is used together with the well known Finite difference numerical method to solve the Stokes problem over a rectangular domain as well as in irregular domain. The iteration of boundary conditions method has been applied to the Stokes problem in a rectangular domain, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $-\frac{d\pi}{2} < y < \frac{d\pi}{2}$, by the above mentioned researchers. Our main task here is to validate the results of the approximate methods by this analytical method in case of the rectangular domain and extend that to the case of irregular domain. The (Chizhonkov and Kargin, 2006) algorithm is typically the best choice for validation purposes because of its high accuracy.

It is known in literature that increasing the parameter d , which represents the ratio of the sides, leads to slow down in convergence of the approximate methods like the conjugate Gradients of Uzawa (Kobelkov and Olshanskii, 2000). It is therefore important that an algorithm that converges uniformly with respect to the parameter d is considered. The (Chizhonkov and Kargin, 2006) algorithm is typical of such an algorithm, and hence our choice of the method in this work.

In this project the non-homogeneous Stokes problem is transformed into a homogeneous Stokes problem and the resulting problem is then decomposed into two sub problems that are solvable by the eigenfunction expansion method. Once all necessary coefficients of the generalised Fourier series are known and the functions describing the boundary conditions are prescribed and represented in terms of the Fourier series, we then proceed to formulate the iteration of boundary conditions numerical algorithm. Finally we develop a numerical scheme, using the finite difference methods, for solving the problem in both rectangular and irregular domains. Coding of the numerical algorithm is done using MATLAB 9.0,R2016a programming language, and implemented by the author. The results of the two methods in both cases of boundary conditions are then compared for validation of our purely numerical results.

Declaration

I, Ndivhuwo Ndou, declare that this thesis titled, 'Numerical Simulations of Stokes Flow by the Iterations of Boundary Conditions and Finite Difference Methods' and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

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Chapter 1

Background of the problem

1.1 Introduction

We considered the Stokes problem in two dimensional space. In general the numerical approach to solving the Stokes problem is divided into three categories. They are Vorticity-Stream function approach, stream function approach only and primitive variable approach where in the velocities and pressure are used simultaneously. Each approach is found to have its own strength over the other in solving Stokes problem and should be applied based on the nature of the problem being solved. The primitive variable approach is known to produce only second order equations with boundary conditions which can be applied directly as opposed to Vorticity-Stream function approach which represents a set of coupled non-linear second order equations which must be solved by using an iteration scheme to obtain a solution. Two types of boundary conditions are suitable here, that is the Dirichlet and Neumann boundary conditions. In this problem the stream function has two boundary conditions while vorticity function has none. The stream function vorticity approach is considered to take into account the non-linear fourth order equation and also requires an iterative techniques to find solutions. However in this project we decided to follow the vorticity Stream function approach for solving our problem as is considered to be suitable when solving unsteady, incompressible Stokes algorithms.

This work seeks to apply the iteration of boundary conditions method to validate the results of the finite difference numerical method carried out in the case of rectangular and irregular domain. A comparison of flow properties like velocity profile, pressure distribution and stream lines produced by the proposed two methods above will be conducted. An agreement of those results will be interpreted to mean that the numerical results have been validated since the analytical method is known to produce excellent results. We have conducted this work in three stages.

The first stage of this work was to implement the analytical method through visualisation and interpretation of the results in the case of a rectangular domains. A MATLAB program was developed to achieve this objective. MATLAB (matrix laboratory) is a multi-paradigm numerical computing environment. It is a proprietary programming language developed by MathWorks. It allows matrix manipulations, plotting of functions and data, implementation of algorithms, creation of user interfaces, and interfacing with programs written in other languages, including C, C++, Java, Fortran and Python [Demuth and Beale \(2009\)](#).

The second stage involves the extension of the analytical method to the case of a domain with the bottom boundary having a curved part. Flow properties were then computed for this case through a MATLAB code we developed.

The last phase of this work is concerned with the development of a numerical scheme and its implementation through MATLAB programs, in the case of a domain with an irregular boundary. The results were then compared with those of the analytical method in order to ascertain the validity of numerical results. It must be noted that the analytical method remains valid even in the case of an irregular domain since the new boundaries conform to all requirements of regularity of the solution to the problem with such boundaries.

1.2 An overview of Navier-Stokes equations

The Stokes problem is derived from the Navier-Stokes equations which is an equation used in the construction of models describing fluid flows (Bachelor, 1967). In particular this equations are nonlinear. Its linear forms are used to model air flow, weather patterns, ocean currents, blood flow in veins, fluid flow in pipes and many other examples. The Navier-Stokes equations are sometimes investigated for purely mathematical reasons. It is well known fact that well-posedness of partial differential equations (PDEs) is key to the construction of meaningful models and this entails proving existence and uniqueness of solutions to those equations (Bachelor, 1967). It is worth noting here that the problem of proving existence and uniqueness of the solution to the Navier-Stokes equation in 3-dimensions is still pending. The Navier-Stokes equation is based on continuum mechanics and is solved for fluid velocity as opposed to position at any given time (Djidjeli et al., 1993). Fluid mechanics is a subject that deals with fluids based on two basic assumptions; i.e. the fluid at rest, known as fluid statics, and the fluid in motion in the case of fluid dynamics (Bachelor, 1967). The study of fluid motion is categorized into Newtonian and Non-Newtonian fluids depending on fluid properties such as viscosity, surface tension and so on. Many scientists have been involved in the study of fluid flows and some of the names of such people are Bernoulli, Stokes, Navier, Cauchy, Hooke, Newton and many others.

1.3 Properties of the Navier-Stokes Equations

The Navier-Stokes equation is a nonlinear partial differential equations describing real life fluid flows. The convective acceleration of fluid particles, which is the change in velocity with respect to time, is the main cause of the nonlinearity effects. As we all have at some stage experienced real life situations of flows, we definitely have observed some chaotic behaviour of many fluid flows. The Reynolds numbers for Newtonian fluids determines the type of flow. Navier-Stokes equations, arguably, represent the most applied fluid motion models and can be extended to cover turbulence as well. In significantly small scales or under severe fluid motion conditions, real fluid flows may have to be looked at in the form of discrete molecules rather than a continuum. However, this work is not extended to cover that.

1.4 Incompressible fluid flow of Newtonian fluids

A Newtonian fluid is understood to be a fluid whose shear stress is linearly proportional to the fluid velocity gradient in the direction perpendicular to the plane of shear (Davis, 1991). Fluids that agree with Newtonian law of viscosity are termed Newtonian fluids; that is water, gases, oil, etc.

In the case of incompressible flows the Navier-Stokes equation is defined by where the terms of the equation represent, respectively, the unsteady acceleration, convective acceleration, pressure gradient and body forces. To complete the description of the flow, the continuity equation is also prescribed in the case where temperature is not included.

1.5 Stokes Flows

The Stokes flows take place at low Reynolds numbers $Re \ll 1$, and are described by linearised Navier-Stokes equations (Nakanishi and Kida, 1999). A lot of studies have been done around the type of flow. There have been studies about flows around stationary as well as moving objects (Hishida and Toshiaki, 2006). Flows around cylindrical and spherical shapes are some of the examples of studies that have been extensively carried out (Tomotika and Aoi, 1950). The Stokes paradox, others call it Jeffery paradox, is one interesting phenomenon in as far as Stokes flows are concerned. It is stated that there does not exist a global solution to the Stokes problem past a cylinder that satisfies the no-slip condition on the cylinder and zero condition at infinity. However our case is the one described for flows in a bounded region.

For potential flows we derive the equation of creeping flow (Stokes flow) through combining continuity equation and irrotationality condition of the flow field (Sisavath et al., 2002). There are many applications of this theory in engineering, medical application and other setups. Stokes' analysis of fluid flows has been used for a very long time now in the construction of models that describe, among other phenomenon, lubrication, flow of micro-organisms and many other examples (Lauga and Powers, 2009). In this work we are concerned with the application of a powerful method called iteration of boundary conditions in solving a general Stokes problem on irregular domain. It is hoped that if the numerical approach to the solution

of this problem is validated then researchers in various disciplines which apply this theory may adopt the numerical method of solution of the problem which is less costly to use.

1.6 Derivation of Navier Stokes Equation

The Navier-Stokes equation is derived from Newton's 2nd law of motion taking into consideration the fluid stress due to viscosity and pressure that develops in the flow.

We first consider the assumption made in Reynold's transport theorem which is defined as follows

where ρ is the density, V is the volume, v is the velocity, A is the area and t is time. The left handside term defines the change in the density over time.

$\int_{\Omega} \rho v \cdot n dA$ shows how much the property P is leaving the given volume.

$\int_{\Omega} Q dV$ shows how much the property p is leaving the given region.

We need to express the flux term in terms of volume integral by using divergence theorem. thus

$$\int_{\delta\Omega} P v \cdot dA = \int_{\Omega} \nabla \cdot (P v) dV \quad (1.1)$$

This implies that

$$\int_{\delta\Omega} P v \cdot dA = - \int_{\Omega} \nabla \cdot (P v) + Q dV \quad (1.2)$$

Thus the above equation can be expressed as follows

$$\int_{\delta\Omega} P v \cdot dA = - \int_{\Omega} \nabla \cdot (P v) + Q dV \quad (1.3)$$

thus

$$\int_{\delta\Omega} P v \cdot dA + \int_{\Omega} \nabla \cdot (P v) + Q dV = 0 \quad (1.4)$$

hence

$$\frac{d\rho}{dt} + \nabla \cdot (\rho v) + Q = 0 \quad (1.5)$$

Due to the fact that the region were the fluid is flowing is constant across the board, this suggest that $Q = 0$ thus

$$\frac{d\rho}{dt} + \nabla \cdot (\rho v) = 0 \quad (1.6)$$

this is the equation of conservation of mass.

we'll need to also employ the concept of material derivative which is necessary for derivation of momentum equation. Thus material derivative is defined as follows

$$\frac{Du}{Dt} = \frac{du}{dt} + (v \cdot \nabla u) \quad (1.7)$$

where $v \cdot \nabla$ symbolised the directional derivative of u in the direction of velocity v

Momentum equation is derived from basic Newton's law defined as follows

$$F = ma$$

By transforming the above equation we obtain the following

$$\rho \frac{dv}{dt}(x, y, z, t) = F \quad (1.8)$$

thus

$$\rho \left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial v}{\partial z} \cdot \frac{\partial z}{\partial t} \right) = F \quad (1.9)$$

Similarly

$$\rho \left(\frac{\partial v}{\partial t} + v \cdot \nabla v \right) = F \quad (1.10)$$

By using the definition of material derivative we then obtain the following expression

$$\rho \frac{\Delta v}{\Delta t} = F \quad (1.11)$$

By assuming that the body force on the fluid is due to fluid stress and external forces, implies the following

$$F = \nabla \cdot \sigma + f \quad (1.12)$$

Due to the fact that the equation of motion depends on stress tensor σ which is defined as follows

The above expression can be expressed as the following

By denoting the stress deviator tensor as S , thus

$$\rho \frac{\Delta v}{\Delta t} = -\nabla p + \nabla \cdot S + f \quad (1.13)$$

For the newtonian and Non-Newtonian fluid the stress is considered as proportional to the rate of change in velocity in the direction of the stress, thus

$$\tau_{i,j} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (1.14)$$

thus for

$$\begin{aligned} (\nabla \sigma)_i &= \mu \frac{\partial}{\partial x} \left(2 \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ &= \mu \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial z} \\ &= \mu \nabla^2 \mathbf{u} + \mu \frac{\partial}{\partial x} (\nabla \cdot \mathbf{v}) \end{aligned}$$

thus $\nabla \cdot S = \mu \nabla^2 S$ Therefore the general Navier-Stoke's equation is described as follows

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f} \quad (1.15)$$

1.7 Symbols used in the Equations

Table 1.1: For symbols in equations and name assigned

Symbols in Equations	Names of the symbols
u, v, u^*, v^*	Velocity of the fluid flow along x and y direction
p, p^*	Pressure distribution
l	Length
t	Time
F_B	Body forces acting in the fluid
T	Tangential velocity

1.8 Greek Symbols

Table 1.2: For Greek symbols and names assigned

Symbols	Names of the symbols
Re	Reynold number
ρ	Density of the fluid
μ	Viscosity
ψ	Stream function
$\nabla \cdot$	Divergence
∇	Gradient
Δ	Laplace
\int	Integration
$\frac{d}{dx}$	Derivation

1.9 The aim and objectives of the study

The aim of this study is to apply the iteration of boundary conditions method to validate the results obtained using the finite difference method in a case of rectangular and irregular domains.

The objectives of the research project are stated below:

In this project we used the iteration of boundary conditions method and well known Finite difference method to solve the Stokes problem in a rectangular domain as well as in a irregular domain.

The finite difference numerical method results are tested against the analytical results in the two cases of a rectangular and irregular domains.

For purposes of illustration of the results, Matlab programmes are developed for the generation of data and graphs of some key fluid properties in both the analytical and numerical cases.

Chapter 2

Literature review

The Stokes problem is governed by the linearized Navier-Stokes equation defined by

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \Delta \mathbf{u} + \mathbf{F}_B \quad (2.1)$$

The Stokes problems describes viscous, incompressible fluid flows in various flow domains. Examples are flows in porous media studied extensively by the likes of (Padmaathi et al., 1998) and many other authors. Problems in this area involve flows around obstacles, in pipes, over plates, past circular cavity and many others. Cavitation problems as described by Stokes problems are good candidates for the application of the iteration method being discussed here. Those Stokes problems modelling various processes have been solved using various methods. Integral equations have been used, numerical methods are widely used and analytical methods are also in use though in a limited scope because of the nature of equations describing real life situations which turn out to be nonlinear rendering the use of analytical methods difficult. The likes of (Cumsille and Tucsnak, 2006) have successfully worked on analytical solutions of Stokes problems with relatively small Reynolds numbers involving rotating and translating bodies in the flow domain. These authors considered and proved global in time and local in time existence and uniqueness respectively, the two-dimensional and three-dimensional problems of a rotating circularly cylindrical body about its axis, in the vertical direction, governed by classical Navier-Stokes equations.

To carry out the proofs, the classical Navier-Stokes equations are initially linearized to a Stokes system. Other authors like (Hishida and Toshiaki, 2006), (Fujita and Kato, 1964) and (Galdi, 1993) have also produced remarkable results on well-posedness of the solution to the problem of a rotating cylindrical body about a vertical axis at low Reynolds numbers. We also make mention the work of (Ueda et al., 2001) on steady viscous flow past two infinite rotating circular cylinders, which also is an example of Stokes flow problems. Like in our case, the solutions to those problems have to be found in special function spaces as opposed to the usual classical solutions in L_2 spaces. Numerical methods have also been extensively used in determining solutions for incompressible viscous flows involving complex flow dynamics. Some of those methods utilize integral equations, for example the work of (Wu et al., 2005) and others.

Others have reduced differential equations into linear algebraic systems and used iterative solvers for the system of linearized unsteady, viscous incompressible Navier-Stokes equations. Olshanskii (2002) is one mathematician who uses this approach with good results

for various sizes of viscosity in both two- and three-dimensional problems. However, it is known that some commonly used numerical methods, like conjugate gradients methods of Uzawa and others, develop some problems in terms of rate of convergence. It is essential that the formulated algorithms are uniformly convergent. Many numerical approaches suffer from lack of uniform convergence and as a result efficiency of such schemes is compromised. Further examples of computational methods can be found in paper on numerical solution of saddle point problems (Benz et al., 2005) as well as paper on relaxation methods on saddle point problems (Chizhonkov, 2002). Taking advantage of the fact that the problem under study is known to have a unique solution, it is common procedure to use the formulation of the problem for justification of numerical solutions presented for the study of Stokes problems.

The iteration method, which is considered to be partial analytical and numerical method, is versatile in the sense that it converges uniformly with respect to the increasing value of the parameter d and is not difficult to implement. (Kobelkov and Olshanskii, 2000) also considered and developed a uniformly convergent numerical scheme for solving nonstationary Stokes equations by making use of a specially developed preconditioner for the Schur complement. Other successful methods are those of Laplace-Beltrami whose approach is based on splitting the boundary conditions, see (Froyd et al., 2013). As opposed to numerical methods, (Chizhonkov and Kargin, 2006) method is analytical, and uses an iteration of the boundary conditions to step forward the iterations. In other words, this is a problem for determination of the solution in parametrically extended domains. The constructed auxiliary problems are solved sequentially to produce the required solution. The trace of the solution to the first auxiliary problem is evaluated at the next prescribed boundaries to produce boundary conditions for the second auxiliary problem. The process is continued for the required number of steps. Adding those pieces gives us the solution to our original problem. Our understanding is that, the method may prove to be easy to apply in, for example, engineering situations better than many other available methods. It is envisaged that after this study the method will be put to test in one of the many situations of applications of the Stokes problems.

The Stokes problems are obtained from Navier-Stokes equations as a result of both linearization and non-dimensionalization of those equations (John et al., 2006). At the level of this study, it is probably crucial to do the non-dimensionalization of the equations in detail. It is well known that non-dimensionalization of an equation means to transform the equation into dimensionless form (Hui, 1987). There are several reasons for doing that. In some cases we end up with reduced number of parameters and terms of the original equations. Non-dimensionalization can also help in the linearization of the equations when comparisons between large and small terms are made as well as reveal some important physical properties of the flow. Before one can do this it is important to understand properly the contributions of each term of the equations so that the scaling can be done to maximize possibilities of obtaining simplified equations. In our case we shall utilize our knowledge that the Stokes problems under study represent highly viscous flows, low Reynolds number flows, to non-dimensionalize our equations. In that way we will use viscous forces to non-dimensionalize our equations instead, for example, of inertia forces. This is book work information from any standard text book of fluid dynamics, however for the purposes of completeness of our work.

Chapter 3

Problem Formulation

3.1 Non-dimensionalisation of Navier-Stokes Equations

The Navier-Stokes equations are non-dimensionalized below term by term. From conservation of momentum equation we have the equation

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \Delta \mathbf{u} + \mathbf{F}_B. \quad (3.1)$$

The above equation can be non-dimensionalized through selection of the appropriate scales as follows

Table 3.1: For dimensionless parameters and names assigned

Scales	Dimensionless variables
Length, l	$x = lx^*, y = ly^*, z = lz^*$
Velocity, U	$\mathbf{u} = U\mathbf{u}^*$
Time,	$\frac{lt^*}{U}$
Operators	$\nabla = l\nabla^*, \Delta = l^2\Delta^*$
Pressure, p	$p = \frac{\mu U p^*}{l}$

We substitute the representations for the variables in the above equation 3.1 to obtain the following non-dimensional terms of the Navier-Stokes equation. Thus the left hand side of the equation becomes

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = \frac{\rho U^2}{l} \frac{\partial \mathbf{u}^*}{\partial t^*} + \frac{\rho U^2}{l} \mathbf{u}^* \cdot \nabla^* \mathbf{u}^*. \quad (3.2)$$

whereas the right handside simplifies to

$$-\nabla p + \mu \Delta \mathbf{u} + \mathbf{F}_B = -\frac{\mu U}{l^2} \nabla^* p^* + \frac{\mu U}{l^2} \Delta^* \mathbf{u}^* + \mathbf{F}_B. \quad (3.3)$$

By equating 3.2 and 3.3 we obtain

$$\frac{\rho U^2}{l} \frac{\partial \mathbf{u}^*}{\partial t^*} + \frac{\rho U^2}{l} \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* = -\frac{\mu U}{l^2} \nabla^* p^* + \frac{\mu U}{l^2} \Delta^* \mathbf{u}^* + \mathbf{F}_B. \quad (3.4)$$

Further, we divide both sides of 3.4 by $\frac{\mu U}{l^2}$ to obtain

$$\frac{\rho U l}{\mu} \frac{\partial \mathbf{u}^*}{\partial t^*} + \frac{\rho U l}{\mu} \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* = -\nabla^* p^* + \Delta^* \mathbf{u}^* + \frac{l^2}{\mu U} \mathbf{F}_B. \quad (3.5)$$

This equation reveals some important features on scaling of the equations as we readily see the reynold number, $Re = \frac{\rho U l}{\mu}$, appearing in some terms of the equation. Thus 3.5 becomes

$$Re \frac{\partial \mathbf{u}^*}{\partial t^*} + Re \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* = -\nabla^* p^* + \Delta^* \mathbf{u}^* + \mathbf{f}. \quad (3.6)$$

Where $\mathbf{f} = \frac{l^2}{\mu U} \mathbf{F}_B$

In our case, the inertia forces are regarded as smaller than the viscous forces, i.e. $Re \rightarrow 0$. In that case the inertia terms can be neglected, leaving the equation of the fluid motion to be described by the equation

$$-\nabla^* \mathbf{u}^* + \nabla^* p^* = \mathbf{f}. \quad (3.7)$$

which is called Stokes equation.

3.2 Stokes problem

Having non-dimensionalized our equations we now are ready to consider the Stokes problem described by the following equations

$$\begin{cases} -\nabla \mathbf{u} + \nabla q = \mathbf{f} & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = 0 & \text{in } \partial\Omega \end{cases} \quad (3.8)$$

The problem is studied in the rectangular domain Ω bounded by the intervals described by

$$-\frac{\pi}{2} < x < \frac{\pi}{2}, -\frac{d\pi}{2} < y < \frac{d\pi}{2}.$$

where parameter d plays a crucial role in the analysis of the flow.

For this problem, simple as it looks, we may not be able to apply the method of iteration of boundary conditions directly. It is then required that we transform it into a form susceptible to the use of the proposed method. This problem is transformed into homogenous Stokes problem by using the solution of the well-known Stokes problem with periodically extended boundary conditions governed by the following equations

$$\begin{cases} -\nabla \mathbf{v} + \nabla r = \mathbf{f} & \text{in } \Omega \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega \\ \mathbf{v} \cdot \mathbf{n} = 0, \frac{\partial(v, \tilde{n})}{\partial n} = 0 & \text{in } \partial\Omega \end{cases} \quad (3.9)$$

The transformed Stokes problem takes the form

$$\begin{cases} -\nabla \mathbf{w} + \nabla p = \mathbf{0} & \text{in } \Omega \\ \nabla \cdot \mathbf{w} = 0 & \text{in } \Omega \\ \mathbf{w} \cdot \mathbf{n} = 0, \mathbf{w} \cdot \boldsymbol{\tau} = 0 & \text{in } \partial\Omega \end{cases} \quad (3.10)$$

where τ the given tangential velocity along the boundary and b , and its fractional derivative of order $\frac{1}{2}$, is a functions in L_2 representing a trace of v along the boundary.

The new vector $\mathbf{w} = \mathbf{v} - \mathbf{u}$ and the pressure function $p = r - q$ are used to transform the partial differential equation in 3.9 into a homogeneous equation given in 3.10. The new problem, 3.10, obtained is sometimes called the generalized cavity problem.

Further, we use linearity of 3.10 to obtain two sub-problems. one with the given tangent on the horizontal and the other with the given tangent on the vertical boundary. We therefore can break up our problem 3.10 into two problems satisfying the same partial differential equation by setting

$$\mathbf{w} = \mathbf{u} + \tilde{\mathbf{u}}, p = \tilde{p} + \tilde{\tilde{p}}, b = a + \tilde{a} \quad (3.11)$$

$$a(x) = \begin{cases} a_1(x), & y = -\frac{d\pi}{2} \\ a_2(x), & y = \frac{d\pi}{2} \end{cases} \quad (3.12)$$

$$\tilde{a}(x) = \begin{cases} \tilde{a}_1(x), & y = -\frac{\pi}{2} \\ \tilde{a}_2(x), & y = \frac{\pi}{2} \end{cases} \quad (3.13)$$

The two new problems, are in the unknown variables u and \tilde{p} , and u and $\tilde{\tilde{p}}$. The corresponding boundary conditions for the two new problems are determined as shown below.

3.3 Boundary conditions

3.3.1 Rectangular domain case.

Boundary conditions $\mathbf{w} \cdot \mathbf{n} = 0$

Let $\mathbf{w} = \mathbf{u} + \tilde{\mathbf{u}}$ and determine the boundary conditions on \mathbf{u} and $\tilde{\mathbf{u}}$ as follows

On the right hand side of the vertical boundary

$$\begin{aligned} \mathbf{w} \cdot \mathbf{n} &= (\mathbf{u} + \tilde{\mathbf{u}})(1, 0) = (u(x), v(y)) + (\tilde{u}(x), \tilde{v}(y)) \cdot (1, 0), \\ &= u(x) + \tilde{u}(x) = 0. \end{aligned} \quad (3.14)$$

Thus the boundary conditions take the form

$$u(x)|_{\partial\Omega_{ver}} = 0, \tilde{u}(x)|_{\partial\Omega_{ver}} = 0. \quad (3.15)$$

On the left hand side of the vertical boundary

$$\mathbf{w} \cdot \mathbf{n} = (\mathbf{u} + \tilde{\mathbf{u}}) \cdot (-1, 0) = (u(x), v(y)) + (\tilde{u}(x), \tilde{v}(y)) \cdot (-1, 0), \quad (3.16)$$

$$= -u(x) - \tilde{u}(x) = 0,$$

$$u(x)|_{\partial\Omega_{ver}} = 0, \tilde{u}(x)|_{\partial\Omega_{ver}} = 0. \quad (3.17)$$

On the upper side of the horizontal boundary

$$\mathbf{w} \cdot \mathbf{n} = (\mathbf{u} + \tilde{\mathbf{u}}) \cdot (0, 1) = (u(x), v(y)) + (\tilde{u}(x), \tilde{v}(y)) \cdot (0, 1), \quad (3.18)$$

$$= v(x) + \tilde{v}(x) = 0.$$

Boundary conditions then take the form

$$v(x)|_{\partial\Omega_{hor}} = 0, \tilde{v}(x)|_{\partial\Omega_{hor}} = 0. \quad (3.19)$$

On the lower side of horizontal boundary

$$\mathbf{w} \cdot \mathbf{n} = (\mathbf{u} + \tilde{\mathbf{u}}) \cdot (0, -1) = (u(x), v(y)) + (\tilde{u}(x), \tilde{v}(y)) \cdot (0, -1), \quad (3.20)$$

$$= -v(x) - \tilde{v}(x) = 0. \quad (3.21)$$

Boundary conditions $\mathbf{w} \cdot \boldsymbol{\tau} = \mathbf{b}$

Also, for the condition $\mathbf{w} \cdot \mathbf{n} = 0$ we use $\mathbf{w} = \mathbf{u} + \tilde{\mathbf{u}}$ to obtain the boundary conditions on \mathbf{u} and $\tilde{\mathbf{u}}$ as follows

On the right hand side vertical boundary

$$(\mathbf{u} + \tilde{\mathbf{u}}) \cdot (0, 1) = (u(x), v(y)) + (\tilde{u}(x), \tilde{v}(y)) \cdot (0, 1) = a + \tilde{a}, \quad (3.22)$$

$$= \tilde{b}(y),$$

$$v(x) + \tilde{v}(x) = \tilde{b}(y). \quad (3.23)$$

hence the boundary conditions take the form

$$v(y)|_{\partial\Omega_{ver}} = 0, \tilde{v}(y)|_{\partial\Omega_{ver}} = \tilde{b}(y) \quad (3.24)$$

On the left hand side of the vertical boundary

$$(\mathbf{u} + \tilde{\mathbf{u}}) \cdot (0, -1) = (u(x), v(y)) + (\tilde{u}(x), \tilde{v}(y)) \cdot (0, -1) = \tilde{a} = \tilde{b}(y). \quad (3.25)$$

$$-v(x) - \tilde{v}(x) = \tilde{a} + \tilde{a}(y), \quad (3.26)$$

$$v(y) = 0, \tilde{v}(y) = \tilde{a}(y). \quad (3.27)$$

Boundary conditions then take the form

$$v(y)|_{\partial\Omega_{ver}} = 0, \tilde{v}(y)|_{\partial\Omega_{ver}} = \tilde{a}(y). \quad (3.28)$$

On the lower side of horizontal boundary

$$\mathbf{w} \cdot \boldsymbol{\tau} = (u(x), v(y)) \cdot (-1, 0) + (\tilde{u}(x), \tilde{v}(y)) \cdot (-1, 0), \quad (3.29)$$

$$-u(x) - \tilde{u}(x) = b(x), \quad (3.30)$$

$$\tilde{u}(x) = 0, u(x) = -b(x). \quad (3.31)$$

Boundary conditions take the form

$$\tilde{u}|_{\partial\Omega_{hor}} = 0, u|_{\partial\Omega_{hor}} = -b(x). \quad (3.32)$$

On the upper side of horizontal boundary

$$\mathbf{w} \cdot \boldsymbol{\tau} = (u(x), v(y)) \cdot (1, 0) + (\tilde{u}(x), \tilde{v}(y)) \cdot (1, 0), \quad (3.33)$$

$$u(x) + \tilde{u}(x) = b(x) = a(x), \quad (3.34)$$

$$u|_{\partial\Omega_{hor}} = -a(x), \tilde{u}|_{\partial\Omega_{hor}} = 0. \quad (3.35)$$

3.4 Boundary conditions of an irregular domain

3.4.1 Domain with irregular boundaries.

The only changes here are at the bottom of the domain where the boundary is made up of straight lines and a curve, which is semi-circular. This semi-circular part of the domain is inserted at the middle of the bottom side of the domain. Below we describe the conditions on this boundary. The conditions on the other boundaries remain unchanged.

At the bottom of the domain the new conditions are given as follows

$$\text{when } x = -\frac{\pi}{2} \text{ to } -\frac{\pi}{4} \text{ and } x = \frac{\pi}{4} \text{ to } \frac{\pi}{2} \text{ for } y = -\frac{d\pi}{2}$$

$$u|_{\partial\Omega_{hor}} = 0, \tilde{u}|_{\partial\Omega_{hor}} = 0 \quad (3.36)$$

when $x = \sqrt{r^2 - (y - y_0)^2} + x_0$ at the semi-circular part of the domain is inserted at the middle of the bottom side of the domain from $x = -\frac{\pi}{4}$ to $\frac{\pi}{4}$, where r is the radius of the semi-circle, (x_0, y_0) is centre coordinates.

$$u|_{\partial\Omega_{hor}} = 0, \tilde{u}|_{\partial\Omega_{hor}} = 0. \quad (3.37)$$

Taking into consideration the above boundary conditions, the new problems are given by

$$\begin{cases} -\nabla \mathbf{u} + \nabla \tilde{p} = \mathbf{f} & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \\ u|_{\partial\Omega_{ver}} = 0, v|_{\partial\Omega} = 0, u|_{\partial\Omega_{hor}} = a(x) \end{cases} \quad (3.38)$$

$$\begin{cases} -\nabla \tilde{\mathbf{u}} + \nabla \tilde{p} = \mathbf{f} & \text{in } \Omega \\ \nabla \cdot \tilde{\mathbf{u}} = 0 & \text{in } \Omega \\ \tilde{u}|_{\partial\Omega_{hor}} = 0, \tilde{u}|_{\partial\Omega} = 0, \tilde{v}|_{\partial\Omega_{ver}} = \tilde{a}(y) \end{cases} \quad (3.39)$$

where $\mathbf{u} = (u, v)$ and $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v})$

The key problem is therefore to solve 3.38 since the solution to 3.9 is known from literature and that 3.39 is obtained from 3.38 through rotating the domain of the flow of the problem by 90° . To do that we formulate two auxiliary problems subject to specially selected boundary conditions. The problems are of the form

$$\begin{cases} -\nabla \mathbf{u} + \nabla \tilde{p} = \mathbf{f} & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \\ u|_{\partial\Omega_{ver}} = 0, u|_{\partial\Omega_{hor}} = a \\ \frac{\partial v}{\partial x}|_{\partial\Omega_{ver}} = 0, v|_{\partial\Omega_{hor}} = 0 \end{cases} \quad (3.40)$$

and

$$\begin{cases} -\nabla \tilde{\mathbf{u}} + \nabla \tilde{p} = \mathbf{f} & \text{in } \Omega \\ \nabla \cdot \tilde{\mathbf{u}} = 0 & \text{in } \Omega \\ \frac{\partial \tilde{u}}{\partial x} |_{\partial \Omega_{hor}} = 0, \tilde{u} |_{\partial \Omega_{ver}} = 0 \\ \tilde{v} |_{\partial \Omega_{hor}} = 0, \tilde{v} |_{\partial \Omega_{ver}} = \tilde{a} \end{cases} \quad (3.41)$$

Problem 3.8 is rendered solved once the solutions for 3.40 and 3.41 are obtained.

Chapter 4

Analytical Method for Solving Stokes problem

Clearly, the Stokes problems are only an approximation of problems involving fully nonlinear equations described by Navier-Stokes equations. Key aspects of the Stokes Problems involve approximations, study of non-local effects and vorticity. The approximation of Navier-Stokes equations are based on scaling analysis and non-dimensionalisation of variables. Depending on the order of forms in the equations some terms may be neglected leading to simple problems which normally are solved without hitches. However, it must be noted that when we neglect some terms the resulting physical properties of the flows for the simplified version may no longer be similar to the initial phenomena.

It is therefore critical to know that the approximations may only correspond to severe situations. With that in mind we shall pursue the approximated solution of the Stokes problem by the iteration method and Finite difference methods cognisant of the fact that the results for these cases cannot be taken blindly into the more complex processes of practical interest. However, those results can help us to make meaningful approximations to certain phenomena.

The nonhomogenous Stokes problem will be transformed into homogeneous Stokes problem and then decomposed into two solvable problems by the eigenfunction expansion method. It will be shown how Stokes problem is deduced from Navier-Stokes equation together with suitable boundary conditions. The flow equations are then non-dimensionalised through a selection of appropriate scales. We chose to deal with a problem where the inertia forces are small compared to viscous forces, that is at small Reynolds number. When the inertia terms are neglected and a creeping flow is obtained.

The Stokes problem obtained will then be solved through an iteration method. To make progress the initial problem was split into two auxiliary problems whose solutions are added up to produce the general solution of the problem. Eigen-function expansion method would be used to solve the auxiliary problems in terms of the Fourier series. For the solution of original problem a scheme of iteration of the boundary conditions will then be introduced. Auxiliary problems are constructed for each value of n and the solutions serve as an approximation at that stage of the calculations. One iteration of the algorithm consists of two steps. The most important aspect of this algorithm is that it converges uniformly with respect to the parameter d whereas it is known from literature that conventional schemes of Uzama type

known as conjugate gradient tend to slow down as d increases.

In this section we will be focusing on finding the analytical solution of stokes problem by solving two auxiliary problems generated from our main problem. By means of solving below two auxiliary problems, it'll then imply that Stokes problem is rendered solved.

Below are the two auxiliary problems we suppose to solve.

$$\begin{cases} -\nabla \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \\ u|_{\partial\Omega_{ver}} = 0, u|_{\partial\Omega_{hor}} = a \\ \frac{\partial v}{\partial x}|_{\partial\Omega_{ver}} = 0, v|_{\partial\Omega_{hor}} = 0 \end{cases} \quad (4.1)$$

and

$$\begin{cases} -\nabla \tilde{\mathbf{u}} + \nabla \tilde{p} = \mathbf{f} & \text{in } \Omega \\ \nabla \cdot \tilde{\mathbf{u}} = 0 & \text{in } \Omega \\ \frac{\partial \tilde{u}}{\partial x}|_{\partial\Omega_{hor}} = 0, \tilde{u}|_{\partial\Omega_{ver}} = 0 \\ \tilde{v}|_{\partial\Omega_{hor}} = 0, \tilde{v}|_{\partial\Omega_{ver}} = \tilde{a} \end{cases} \quad (4.2)$$

Equation (3.2) is obtained through rotating the domain of (3.2) by 90° . Our main task now is to solve (2.1) through the well-known method of eigen-function expansion. Equation (2.1) can be written as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x} = 0, \quad (4.3)$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - \frac{\partial p}{\partial y} = 0, \quad (4.4)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (4.5)$$

where $\mathbf{u} = (u, v)$, and this problem is subjected to the following boundary conditions

$$u\left(\frac{\pi}{2}, y\right) = 0, \quad u\left(-\frac{\pi}{2}, y\right) = 0, \quad (4.6)$$

$$u\left(x, \frac{d\pi}{2}\right) = a, \quad u\left(x, -\frac{d\pi}{2}\right) = a, \quad (4.7)$$

$$\frac{\partial v}{\partial x}\left(\frac{\pi}{2}, y\right) = 0, \quad \frac{\partial v}{\partial x}\left(-\frac{\pi}{2}, y\right) = 0, \quad (4.8)$$

$$v\left(x, \frac{d\pi}{2}\right) = 0, \quad u\left(x, -\frac{d\pi}{2}\right) = a. \quad (4.9)$$

To find the solution set $\{u, v, p\}$ we apply eigen-function expansion method. But due to the fact that we are dealing with non-homogenous equation, eigen-function of related homogenous problem is used. Thus in this case we are going to use the solution of homogenous problem below

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \quad (4.10)$$

The eigen-function of the above equation is $\sin\left(x + \frac{\pi}{2}\right)$. The functions u, v and p are assumed to be piecewise smooth such that they can be represented as Fourier series as follows

$$u(x, y) = \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} u_m(y) \sin m\left(x + \frac{\pi}{2}\right), \quad (4.11)$$

$$v(x, y) = \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} v_m(y) \cos m\left(x + \frac{\pi}{2}\right), \quad (4.12)$$

$$p(x, y) = \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} p_m(y) \cos m\left(x + \frac{\pi}{2}\right), \quad (4.13)$$

By assuming $a(y)$ and $b(y)$ are smooth function on $\partial\Omega$. Writing this in terms of fourier series we get the following

$$a(y) = \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} a_m(y) \sin m\left(x + \frac{\pi}{2}\right), \quad b(y) = \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} b_m(y) \cos m\left(x + \frac{\pi}{2}\right) \quad (4.14)$$

Taking advantage of orthogonality of eigenfunctions, the coefficients a_m and b_m are determined from the following formulae

$$a_m(y) = \sqrt{\frac{2}{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a(y) \sin m\left(x + \frac{\pi}{2}\right) \quad (4.15)$$

and

$$b_m(y) = \sqrt{\frac{2}{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b(y) \cos m\left(x + \frac{\pi}{2}\right) \quad (4.16)$$

By substituting (4.11), (4.12), (4.13) into the partial differential equation in (2.1), we get

$$-m^2 \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} u_m(y) \sin m(x + \frac{\pi}{2}) + \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} u_m''(y) \cos m(x + \frac{\pi}{2}) + m \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} p_m(y) \sin m(x + \frac{\pi}{2}) = 0, \quad (4.17)$$

$$-m^2 \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} v_m(y) \sin m(x + \frac{\pi}{2}) + \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} v_m''(y) \cos m(x + \frac{\pi}{2}) - m \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} p_m(y) \sin m(x + \frac{\pi}{2}) = 0, \quad (4.18)$$

$$m \sqrt{\frac{2}{\pi}} u_m(y) \cos m(x + \frac{\pi}{2}) + \sqrt{\frac{2}{\pi}} p_m(y) \cos m(x + \frac{\pi}{2}) = 0. \quad (4.19)$$

Further simplifications of the above equations lead to the following coupled system

A system of this kind has a unique solution, which can be conveniently expressed in terms of hyperbolic functions. To solve the system we proceed as follows

$$\begin{cases} m^2 u_m - u_m'' - m p_m = 0 & (i) \\ m^2 v_m - v_m'' + p_m' = 0 & (ii) \\ m u_m + v_m' = 0 & (iii) \\ u_m \left(\frac{d\pi}{2} \right) = b_m; u_m \left(-\frac{d\pi}{2} \right) = a_m \\ v_m \left(\frac{d\pi}{2} \right) = 0; v_m \left(-\frac{d\pi}{2} \right) = 0. \end{cases} \quad (4.20)$$

By differentiating (iii) once with respect to y we obtain

$$m u_m' + v_m'' = 0 \quad \rightarrow v_m'' = -m u_m'. \quad (4.21)$$

Substitute this into (ii) to obtain

$$m^2 v_m + m u_m' + p_m'' = 0. \quad (4.22)$$

Now differentiate this once with respect to y to get

$$m^2 v_m' + m u_m'' + p_m''' = 0 \quad \text{or} \quad -m^3 u_m + m u_m'' + p_m'' = 0, \quad (4.23)$$

Since

$$v_m' = -m u_m. \quad (4.24)$$

We multiply (i) by m and add the resulting equation to (4.24), to get

$$p_m'' - m^2 p_m = 0. \quad (4.25)$$

Solving for p_m we get

$$p_m = c_1 e^{\alpha y} + c_2 e^{-\alpha y}. \quad (4.26)$$

Substitute (4.26) into (i) to get

$$u''_m - m^2 u_m = -m(c_1 e^{my} + c_2 e^{-my}). \quad (4.27)$$

Now Consider

$$u''_m - m^2 u_m = 0. \quad (4.28)$$

Thus

$$u = C(y) e^{my} + D(y) e^{-my}. \quad (4.29)$$

By method of variation of parameters, we have

$$\begin{aligned} C'(y) e^{my} + D'(y) e^{-my} &= 0, \\ mC'(y) e^{my} - mD'(y) e^{-my} &= -m(C_1 e^{my} + C_2 e^{-my}). \end{aligned}$$

Adding the two equations above, we get first order ordinary differential equation.

$$C'(y) = -\frac{c_1}{2} + \frac{c_2}{2} e^{-my}, \quad (4.30)$$

whose solution is

$$C(y) = -\frac{c_1}{2} y - \frac{c_2}{4m} e^{-2my}. \quad (4.31)$$

Similarly, we solve for $D(y)$ from the equation.

$$D'(y) = \frac{c_1}{2} e^{2my} - \frac{c_2}{2}, \quad (4.32)$$

to obtain

$$D(y) = \frac{c_1}{4m} e^{2my} - \frac{c_2}{2} y. \quad (4.33)$$

Thus

$$\begin{aligned} u_m(y) &= -\left(\frac{c_1}{2} y + \frac{c_2}{4m} e^{-2my}\right) e^{my} + \left(\frac{c_1}{4m} e^{2my} - \frac{c_2}{2} y\right) e^{-my} \\ &= \left(\frac{1}{2m} - y\right) \frac{c_1}{2} e^{my} - \frac{c_2}{2} e^{-my} \left(\frac{1}{2m} + y\right) = \left(\frac{1}{2m} - y\right) \frac{c_1}{2} (\cosh my + \sinh my) \\ &\quad - \frac{c_2}{2} \left(\frac{1}{2m} + y\right) (\cosh my - \sinh my). \end{aligned} \quad (4.34)$$

Since $\cosh y + \sinh y = e^y$ and $\cosh y - \sinh y = e^{-y}$ the expression for u_m then takes the form

$$u_m(y) = \left[\left(\frac{1}{2m} - y \right) \frac{c_1}{2} - \frac{c_2}{2} \left(\frac{1}{2m} + y \right) \right] \cosh m y + \left[\left(\frac{1}{2m} - y \right) \frac{c_1}{2} - \frac{c_2}{2} \left(\frac{1}{2m} + y \right) \right] \sinh m y. \quad (4.35)$$

Now we apply the boundary condition $u_m\left(\frac{d\pi}{2}\right) = b_m$ to obtain the unknown coefficients

$$u_m\left(\frac{d\pi}{2}\right) = \left[\left(\frac{1}{2m} - \frac{d\pi}{2} \right) \frac{c_1}{2} - \frac{c_2}{2} \left(\frac{1}{2m} + \frac{d\pi}{2} \right) \right] \cosh m \frac{d\pi}{2} + \left[\left(\frac{1}{2m} - \frac{d\pi}{2} \right) \frac{c_1}{2} - \frac{c_2}{2} \left(\frac{1}{2m} + \frac{d\pi}{2} \right) \right] \sinh m \frac{d\pi}{2} = b_m, \quad (4.36)$$

$$u_m\left(-\frac{d\pi}{2}\right) = \left[\left(\frac{1}{2m} + \frac{d\pi}{2} \right) \frac{c_1}{2} - \frac{c_2}{2} \left(\frac{1}{2m} - \frac{d\pi}{2} \right) \right] \cosh m \frac{d\pi}{2} + \left[\left(\frac{1}{2m} + \frac{d\pi}{2} \right) \frac{c_1}{2} - \frac{c_2}{2} \left(\frac{1}{2m} - \frac{d\pi}{2} \right) \right] \sinh m \frac{d\pi}{2} = a_m. \quad (4.37)$$

By adding (4.36) and (4.37), we get

$$\left[\frac{1}{2m}(c_1 - c_2) \right] \cosh m \frac{d\pi}{2} + \left[\frac{1}{2m}(c_1 - c_2) \right] \sinh m \frac{d\pi}{2} = a_m + b_m, \quad (4.38)$$

$$c_1 = \frac{2m(a_m + b_m)}{\cosh m \frac{d\pi}{2} + \sinh m \frac{d\pi}{2}} + c_2. \quad (4.39)$$

Substitute (4.39) into (4.35), we get

$$c_2 = -\frac{2}{d\pi} \left[\frac{b_m \left(\frac{3}{2} - m \frac{d\pi}{2} \right) + m \left(\frac{1}{2m} - \frac{d\pi}{2} \right) a_m}{\cosh m \frac{d\pi}{2} + \sinh m \frac{d\pi}{2}} \right]. \quad (4.40)$$

Substitute (4.40) into (4.35), to obtain

$$\begin{aligned}
 u_m(y) = & \left[\frac{1}{2} \left(\frac{1}{2m} - y \right) \frac{a_m \left(2m + \frac{1}{2} - m \frac{d\pi}{2} \right) + b_m \left(2m - \frac{2}{d\pi} \left(\frac{3}{2} - m \frac{d\pi}{2} \right) \right)}{\cosh m \left(\frac{d\pi}{2} \right) + \sinh m \left(\frac{d\pi}{2} \right)} \right. \\
 & + \frac{2}{d\pi} \left(\frac{b_m \left(\frac{3}{2} - m \frac{d\pi}{2} \right) + m \left(\frac{1}{2m} - \frac{d\pi}{2} \right) a_m}{\cosh m \left(\frac{\pi}{2} \right) + \sinh m \left(\frac{d\pi}{2} \right)} \right) \left. \left(\frac{1}{2m} + y \right) \right] \cosh my \\
 & + \frac{1}{2} \left(\frac{1}{2m} - y \right) \frac{a_m \left(2m + \frac{1}{2} - m \frac{d\pi}{2} \right) + b_m \left(2m - \frac{2}{d\pi} \left(\frac{3}{2} - m \frac{d\pi}{2} \right) \right)}{\cosh m \frac{d\pi}{2} + \sinh m \frac{d\pi}{2}} \frac{1}{2} \left(\frac{1}{2m} + y \right) \\
 & + \frac{2}{d\pi} \left(\frac{\frac{b_m \left(\frac{3}{2} - m \frac{d\pi}{2} \right) + m \left(\frac{1}{2m} - \frac{d\pi}{2} \right)}{2} + m \left(\frac{1}{2m} - \frac{d\pi}{2} \right) a_m}{\cosh m \frac{d\pi}{2} + \sinh m \frac{d\pi}{2}} \right) \sinh my. \quad (4.41)
 \end{aligned}$$

Thus we obtain the following

$$\begin{aligned}
 v_m(y) = & \left[\frac{1}{2} \left(\frac{1}{2m^2} - 2m^2 + \frac{y}{m} \right) \frac{a_m \left(2m + \frac{1}{2} - m \frac{d\pi}{2} \right) + b_m \left(2m - \frac{2}{d\pi} \left(\frac{3}{2} - m \frac{d\pi}{2} \right) \right)}{\cosh m \frac{d\pi}{2} + \sinh m \frac{d\pi}{2}} \right. \\
 & + \frac{2}{d\pi} \left[\frac{b_m \left(\frac{3}{2} - m \frac{d\pi}{2} \right) + m \left(\frac{1}{2m} - \frac{d\pi}{2} \right) a_m}{\cosh m \frac{d\pi}{2} + \sinh m \frac{d\pi}{2}} \right] \left(\frac{3}{2m^2} - 2m^2 + \frac{y}{m} \right) \cosh m y \\
 & + \left[\frac{1}{2} \left(\frac{1}{2m^2} - 2m^2 - \frac{y}{m} \right) \frac{a_m \left(2m + \frac{1}{2} - m \frac{d\pi}{2} \right) + b_m \left(2m - \frac{2}{d\pi} \left(\frac{3}{2} - m \frac{d\pi}{2} \right) \right)}{\cosh m \frac{d\pi}{2} + \sinh m \frac{d\pi}{2}} \right. \\
 & \left. - \frac{1}{2} \left(\frac{3}{2m^2} - 2m^2 - \frac{y}{m} \right) \frac{2}{d\pi} \left[\frac{b_m \left(\frac{3}{2} - m \frac{d\pi}{2} \right) + m \left(\frac{1}{2m} - \frac{d\pi}{2} \right) a_m}{\cosh m \frac{d\pi}{2} + \sinh m \frac{d\pi}{2}} \right] \right] \sinh m y. \quad (4.42)
 \end{aligned}$$

and

$$\begin{aligned}
 p_m(y) = & \left(\frac{a_m \left(2m + \frac{1}{2} - m \frac{d\pi}{2} \right) + b_m \left(2m - \frac{2}{d\pi} \left(\frac{3}{2} - m \frac{d\pi}{2} \right) \right)}{\cosh m \frac{d\pi}{2} + \sinh m \frac{d\pi}{2}} \right. \\
 & - \frac{2}{d\pi} \left[\frac{b_m \left(\frac{3}{2} - m \frac{d\pi}{2} \right) + m \left(\frac{1}{2m} - \frac{d\pi}{2} \right) a_m}{\cosh m \frac{d\pi}{2} + \sinh m \frac{d\pi}{2}} \right] \cosh my \\
 & + \left(\frac{a_m \left(2m + \frac{1}{2} - m \frac{d\pi}{2} \right) + b_m \left(2m - \frac{2}{d\pi} \left(\frac{3}{2} - m \frac{d\pi}{2} \right) \right)}{\cosh m \frac{d\pi}{2} + \sinh m \frac{d\pi}{2}} \right. \\
 & \left. + \frac{2}{d\pi} \left[\frac{b_m \left(\frac{3}{2} - m \frac{d\pi}{2} \right) + m \left(\frac{1}{2m} - \frac{d\pi}{2} \right) a_m}{\cosh m \frac{d\pi}{2} + \sinh m \frac{d\pi}{2}} \right] \right) \sinh my. \tag{4.43}
 \end{aligned}$$

Further, we make the following denotations to make the expressions of u_m, v_m and p_m .

$$\left\{ \begin{aligned}
 K_1 &= \frac{b_m - a_m}{2 (\sinh(md\pi) + md\pi)}; K_2 = \frac{b_m + a_m}{2 (\sinh(md\pi) - md\pi)} \\
 R_1 &= md\pi \sinh \left(\frac{md\pi}{2} \right); R_2 = 2 \cosh \left(\frac{md\pi}{2} \right) \\
 S_1 &= md\pi \cosh \left(\frac{md\pi}{2} \right); S_2 = 2 \sinh \left(\frac{md\pi}{2} \right)
 \end{aligned} \right. \tag{4.44}$$

In that way we have

$$\left\{ \begin{aligned}
 u_m(y) &= K_2 [(S_2 - S_1) \cosh my + S_2 (my) \sinh m y] \\
 &+ K_1 [(R_2 - R_1) \sinh m y + R_2 (my) \cosh m y] \\
 v_m(y) &= K_1 [R_1 \cosh m y - R_2 (my) \sinh m y] \\
 &+ K_2 [S_1 \sinh m y - S_2 \cosh m y] \\
 p_m(y) &= -2m [K_2 S_2 \cosh m y + K_1 R_2 \sinh m y]
 \end{aligned} \right. \tag{4.45}$$

For the construction of the iteration process we need to solve the second auxiliary problem. To the first component of \tilde{u} impose the second kind boundary condition as horizontal boundary conditions. Thus

$$\left\{ \begin{aligned}
 -\nabla \mathbf{u} + \nabla p &= \mathbf{f} & \text{in } \Omega \\
 \nabla \cdot \mathbf{u} &= 0 & \text{in } \Omega \\
 u|_{\partial\Omega_{ver}} &= 0, u|_{\partial\Omega_{hor}} = a \\
 \frac{\partial v}{\partial x}|_{\partial\Omega_{ver}} &= 0, v|_{\partial\Omega_{hor}} = 0
 \end{aligned} \right. \tag{4.46}$$

The solution of this problem follows from the steps undertaken in the case of the first auxiliary problem. For this problem

$$\tilde{u} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{u}_m(x) \cosh m \left(\frac{y}{d} + \frac{\pi}{2} \right), \tag{4.47}$$

$$\tilde{v} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{v}_m(x) \sinh m \left(\frac{y}{d} + \frac{\pi}{2} \right), \quad (4.48)$$

$$\tilde{p}_m = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{p}_m(x) \cosh m \left(\frac{y}{d} + \frac{\pi}{2} \right). \quad (4.49)$$

Similarly as in the first auxiliary problem, the coefficients \tilde{u}_m, \tilde{v}_m and \tilde{p}_m are expressed as follows

$$\left\{ \begin{array}{l} \tilde{u}_m = \tilde{K}_1 \left[\tilde{R}_1 \cosh \left(\frac{mx}{d} \right) - \tilde{R}_2 \left(\frac{mx}{d} \right) \sinh \left(\frac{mx}{d} \right) \right] + \tilde{K}_2 \left[\tilde{S}_1 \sinh \left(\frac{mx}{d} \right) - \tilde{S}_2 \left(\frac{mx}{d} \right) \cosh \left(\frac{mx}{d} \right) \right] \\ \tilde{v}_m = \tilde{K}_2 \left[\left(\tilde{S}_2 - \tilde{S}_1 \right) \cosh \left(\frac{mx}{d} \right) + \tilde{S}_2 \left(\frac{mx}{d} \right) \sinh \left(\frac{mx}{d} \right) \right] + \\ \tilde{K}_1 \left[\left(\tilde{R}_2 - \tilde{R}_1 \right) \sinh \left(\frac{mx}{d} \right) + \tilde{R}_2 \left(\frac{mx}{d} \right) \cosh \left(\frac{mx}{d} \right) \right] \tilde{p}_m \\ = -2md \left[\tilde{K}_2 \cosh \left(\frac{mx}{d} \right) + \tilde{K}_1 \tilde{R}_2 \sinh \left(\frac{mx}{d} \right) \right] \end{array} \right. \quad (4.50)$$

where we set

$$\left\{ \begin{array}{l} \tilde{K}_1 = \frac{\tilde{b}_m - \tilde{a}_m}{2 \left(\sinh \left(\frac{m\pi}{d} \right) + \frac{m\pi}{d} \right)}, \quad \tilde{K}_2 = \frac{\tilde{b}_m + \tilde{a}_m}{2 \left(\sinh \left(\frac{m\pi}{2} \right) - \frac{m\pi}{d} \right)} \\ \tilde{R}_1 = \frac{m\pi}{d} \sinh \left(\frac{m\pi}{d} \right), \quad \tilde{R}_2 = 2 \cosh \left(\frac{m\pi}{2} \right) \\ \tilde{S}_1 = \frac{m\pi}{d} \cosh \left(\frac{m\pi}{2d} \right), \quad \tilde{S}_2 = 2 \sinh \left(\frac{m\pi}{2d} \right) \end{array} \right. \quad (4.51)$$

4.1 Method of iteration of boundary conditions

Here we implement the method of iteration of boundary conditions for the approximation of the solution to our Stokes problem. The first auxiliary problem is solved after which its solution is used to calculate the boundary conditions for use in solving the second auxiliary problem. The Fourier series solution to the first auxiliary problem as determined above, is given by

$$u(x, y) = \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} \frac{b_m + a_m}{2 (\sinh (md\pi) - md\pi)} \left[\left(2 \sinh \left(\frac{md\pi}{2} \right) - md\pi \cosh \left(\frac{md\pi}{2} \right) \right) \cosh (my) \right. \\ \left. + 2 \sinh \left(\frac{md\pi}{2} \right) \sinh (my) 2 \cosh \left(\frac{md\pi}{2} \right) (my) \cosh (my) \right] \sin m \left(x + \frac{\pi}{2} \right). \quad (4.52)$$

The boundary condition $u\left(\frac{\pi}{2}, y\right)$ is applied on this solution to obtain

$$\begin{aligned}
 u\left(\frac{\pi}{2}, y\right) &= \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} \frac{b_m + a_m}{2(\sinh(md\pi) - md\pi)} \left[\left(2\sinh\left(\frac{md\pi}{2}\right) - md\pi \cosh\left(\frac{md\pi}{2}\right) \right) \cosh(my) \right. \\
 &\quad \left. + 2\sinh\left(\frac{md\pi}{2}\right) \sinh(my) \right] \sin m\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 0.
 \end{aligned} \tag{4.53}$$

And boundary condition $u\left(-\frac{\pi}{2}, y\right)$ is used, to obtain

$$\begin{aligned}
 u\left(-\frac{\pi}{2}, y\right) &= \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} \frac{b_m + a_m}{2(\sinh(md\pi) - md\pi)} \left[\left(2\sinh\left(\frac{md\pi}{2}\right) - md\pi \cosh\left(\frac{md\pi}{2}\right) \right) \cosh(my) \right. \\
 &\quad \left. + 2\sinh\left(\frac{md\pi}{2}\right) \sinh(my) \right] \sin m\left(-\frac{\pi}{2} + \frac{\pi}{2}\right) = 0.
 \end{aligned} \tag{4.54}$$

Thus the condition on $u(x, y)$ along the vertical boundary is given by

$$u|_{\partial\Omega_{ver}} = 0.$$

Boundary condition $u\left(x, \frac{d\pi}{2}\right)$ is used, to get

$$\begin{aligned}
 u\left(x, \frac{d\pi}{2}\right) &= \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} \frac{b_m + a_m}{2(\sinh(-md\pi) - md\pi)} \left[\left(2\sinh\left(\frac{-md\pi}{2}\right) \right. \right. \\
 &\quad \left. \left. - md\pi \cosh\left(\frac{-md\pi}{2}\right) \right) \cosh\left(m\frac{d\pi}{2}\right) \right. \\
 &\quad \left. + 2\sinh\left(\frac{md\pi}{2}\right) \left(\frac{md\pi}{2}\right) \sinh\left(m\frac{d\pi}{2}\right) \right] \left[\left(\cosh\left(\frac{md\pi}{2}\right) \right. \right. \\
 &\quad \left. \left. - md\pi \sinh\left(\frac{md\pi}{2}\right) \right) \sinh\left(m\frac{d\pi}{2}\right) \right. \\
 &\quad \left. + 2\cosh\left(\frac{md\pi}{2}\right) \left(m\frac{d\pi}{2}\right) \cosh\left(m\frac{d\pi}{2}\right) \right] \sin m\left(x + \frac{\pi}{2}\right) = \alpha(x).
 \end{aligned} \tag{4.55}$$

Also, the boundary condition $u\left(x, -\frac{d\pi}{2}\right)$ is used, to get

$$\begin{aligned}
 u\left(x, -\frac{d\pi}{2}\right) &= \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} \frac{b_m+a_m}{2\left(\sinh\left(-md\pi\right) - md\pi\right)} \left[\left(2\sinh\left(\frac{-md\pi}{2}\right) \right. \right. \\
 &\quad \left. \left. - md\pi \cosh\left(\frac{-md\pi}{2}\right)\right) \cosh\left(m\frac{-d\pi}{2}\right) \right. \\
 &\quad \left. + 2\sinh\left(\frac{md\pi}{2}\right) \left(\frac{md\pi}{2}\right) \sinh\left(m\frac{d\pi}{2}\right) \right] \left[\left(\cosh\left(\frac{md\pi}{2}\right) \right. \right. \\
 &\quad \left. \left. - md\pi \sinh\left(\frac{md\pi}{2}\right)\right) \sinh\left(m\frac{d\pi}{2}\right) \right. \\
 &\quad \left. + 2\cosh\left(\frac{md\pi}{2}\right) \left(m\frac{d\pi}{2}\right) \cosh\left(m\frac{d\pi}{2}\right) \right] \sin m\left(x+\frac{\pi}{2}\right) = \alpha(x). \tag{4.56}
 \end{aligned}$$

Thus the condition on $u(x, y)$ along the horizontal boundary is given by

$$u|_{\partial\Omega_{hor}} = \alpha(x).$$

The boundary conditions on $v(x, y)$ are calculated from its already known expression

$$\begin{aligned}
 v(x, y) &= \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} \frac{b_m-a_m}{2\left(\sinh\left(md\pi\right) + md\pi\right)} \left[md\pi \sinh\left(\frac{md\pi}{2}\right) \cosh\left(my\right) \right. \\
 &\quad \left. - 2\cosh\left(\frac{md\pi}{2}\right) \left(my\right) \sinh\left(my\right) \right] + \frac{b_m+a_m}{2\left(\sinh\left(md\pi\right) - md\pi\right)} \left[md\pi \sinh\left(my\right) \right. \\
 &\quad \left. - 2\sinh\left(\frac{md\pi}{2}\right) \left(my\right) \cosh\left(my\right) \right] \cos m\left(x+\frac{\pi}{2}\right). \tag{4.57}
 \end{aligned}$$

We differentiate the above expression with respect to x, to obtain

$$\begin{aligned}
 \frac{\partial v}{\partial x} &= -m \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} \frac{b_m-a_m}{2\left(\sinh\left(md\pi\right) + md\pi\right)} \left[md\pi \sinh\left(\frac{md\pi}{2}\right) \cosh\left(my\right) \right. \\
 &\quad \left. - 2\cosh\left(\frac{md\pi}{2}\right) \left(my\right) \sinh\left(my\right) \right] + \frac{b_m+a_m}{2\left(\sinh\left(md\pi\right) - md\pi\right)} \left[md\pi \cosh\left(\frac{md\pi}{2}\right) \sinh\left(my\right) \right. \\
 &\quad \left. - 2\sinh\left(\frac{md\pi}{2}\right) \left(my\right) \cosh\left(my\right) \right] \sin m\left(x+\frac{\pi}{2}\right). \tag{4.58}
 \end{aligned}$$

The boundary condition $\frac{\partial v}{\partial x} \left(\frac{\pi}{2}, y \right)$ is used, to obtain

$$\begin{aligned} \frac{\partial v}{\partial x} \left(\frac{\pi}{2}, y \right) = & -m \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} \frac{b_m - a_m}{2 (\sinh (md\pi) + md\pi)} \left[md\pi \sinh \left(\frac{md\pi}{2} \right) \cosh (my) \right. \\ & \left. - 2 \cosh \left(\frac{md\pi}{2} \right) (my) \sinh (my) \right] + \frac{b_m + a_m}{2 (\sinh (md\pi) - md\pi)} \left[md\pi \cosh \left(\frac{md\pi}{2} \right) \sinh (my) \right. \\ & \left. - 2 \sinh \left(\frac{md\pi}{2} \right) (my) \cosh (my) \right] \sin m \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = 0. \end{aligned} \quad (4.59)$$

And the boundary condition $\frac{\partial v}{\partial x} \left(-\frac{\pi}{2}, y \right)$ is used, to obtain

$$\begin{aligned} \frac{\partial v}{\partial x} \left(-\frac{\pi}{2}, y \right) = & -m \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} \frac{b_m - a_m}{2 (\sinh (md\pi) + md\pi)} \left[md\pi \sinh \left(\frac{md\pi}{2} \right) \cosh (my) \right. \\ & \left. - 2 \cosh \left(\frac{md\pi}{2} \right) (my) \sinh (my) \right] + \frac{b_m + a_m}{2 (\sinh (md\pi) - md\pi)} \\ & * \left[md\pi \cosh \left(\frac{md\pi}{2} \right) \sinh (my) - \sinh \left(\frac{md\pi}{2} \right) (my) \cosh (my) \right] \\ & * \sin m \left(-\frac{\pi}{2} + \frac{\pi}{2} \right) = 0. \end{aligned} \quad (4.60)$$

Thus the condition on $\frac{\partial v}{\partial x} (x, y)$ along the vertical boundary is given by

$$\frac{\partial v}{\partial x} \Big|_{\partial \Omega_{ver}} = 0.$$

Also, the boundary condition $v \left(x, \frac{d\pi}{2} \right)$ is used, to get

$$\begin{aligned} v \left(x, \frac{d\pi}{2} \right) = & \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} \frac{b_m - a_m}{2 (\sinh (md\pi) + md\pi)} \left[md\pi \sinh \left(\frac{md\pi}{2} \right) \cosh \left(m \frac{d\pi}{2} \right) \right. \\ & \left. - 2 \cosh \left(\frac{md\pi}{2} \right) \left(m \frac{d\pi}{2} \right) \sinh \left(m \frac{d\pi}{2} \right) \right] + \frac{b_m + a_m}{2 (\sinh (md\pi) - md\pi)} \\ & * \left[md\pi \cosh \left(\frac{md\pi}{2} \right) \sinh (my) - 2 \sinh \left(\frac{md\pi}{2} \right) (my) \cosh (my) \right] \\ & * \cos m \left(x + \frac{\pi}{2} \right) = 0. \end{aligned} \quad (4.61)$$

Whereas the boundary condition $v\left(x, -\frac{d\pi}{2}\right)$ is used, to obtain

$$\begin{aligned}
 v\left(x, -\frac{d\pi}{2}\right) &= \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} \frac{b_m - a_m}{2\left(\sinh\left(md\pi\right) + md\pi\right)} \left[md\pi \sinh\left(\frac{md\pi}{2}\right) \cosh\left(-m\frac{d\pi}{2}\right) \right. \\
 &\quad \left. - 2\cosh\left(\frac{md\pi}{2}\right) \left(-m\frac{d\pi}{2}\right) \sinh\left(-m\frac{d\pi}{2}\right) \right] \\
 &\quad + \frac{b_m + a_m}{2\left(\sinh\left(md\pi\right) - md\pi\right)} \left[md\pi \cosh\left(-\frac{md\pi}{2}\right) \sinh\left(-m\frac{d\pi}{2}\right) \right. \\
 &\quad \left. - 2\sinh\left(-\frac{md\pi}{2}\right) \left(my\right) \cosh\left(my\right) \right] \cos m\left(x + \frac{\pi}{2}\right) = 0
 \end{aligned} \tag{4.62}$$

Thus the condition on $v(x, y)$ along horizontal boundary is given by

$$v|_{\partial\Omega_{hor}} = 0.$$

We obtained new boundary conditions for the first auxiliary problem, which are

$$\begin{cases} u|_{\partial\Omega_{ver}} = 0, u|_{\partial\Omega_{hor}} = \alpha(x) \\ \frac{\partial v}{\partial x}|_{\partial\Omega_{ver}} = 0, v|_{\partial\Omega_{hor}} = 0 \end{cases} \tag{4.63}$$

In the same way we obtain the boundary conditions for the second auxiliary problem from the expressions for the velocity components as calculated above. The resulting boundary conditions are

$$\begin{cases} \frac{\partial \tilde{u}}{\partial y}|_{\partial\Omega_{hor}} = 0, \tilde{u}|_{\partial\Omega_{ver}} = 0 \\ \tilde{v}|_{\partial\Omega_{hor}} = 0, \tilde{v}|_{\partial\Omega_{ver}} = \tilde{a}(y) \end{cases} \tag{4.64}$$

4.2 Iteration algorithm

Iteration method is employed to solve our auxiliary problems. Iteration scheme consist of 2 steps undertaken sequentially by first solving the first auxiliary problem after which we use that solution to obtain boundary conditions to be used in the solution of the second auxiliary problem. The solution of the first auxiliary problem plus the solution of the second auxiliary problem give us the *zeroth* approximation of the solution of our problem. The solution of the second auxiliary problem is then used to provide the boundary conditions for the first auxiliary problem in the subsequent iteration. This first step is towards finding the first iteration or the second term of the sum representing an approximation to the solution of the main problem. After n iterations the solution to our main problem is finally represented in the form

$$\begin{aligned}
 U_1 &= \widehat{U}_1 + \widetilde{U}_1, \\
 U_2 &= \widehat{U}_1 + \widetilde{U}_1 + \widehat{U}_2 + \widetilde{U}_2, \\
 U_n &= \widehat{U}_1 + \widetilde{U}_1 + \widehat{U}_2 + \widetilde{U}_2 + \cdots + \widehat{U}_n + \widetilde{U}_n, \\
 U_n &= \sum_{i=1}^n (\widehat{U}_i + \widetilde{U}_i).
 \end{aligned}$$

Here, we demonstrate how the procedure works. Firstly, we consider the first auxiliary problem with the following boundary conditions

$$\begin{cases}
 u|_{\partial\Omega_{ver}} = 0, & u|_{\partial\Omega_{hor}} = \alpha \\
 \frac{\partial v}{\partial x}|_{\partial\Omega_{ver}} = 0, & v|_{\partial\Omega_{hor}} = 0
 \end{cases}$$

The first approximation to the solution to this auxiliary problem is of the form

$$\begin{aligned}
 \widehat{U}_1 = \left\{ \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} u_m(y) \sin m \left(x + \frac{\pi}{2} \right), \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} v_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right) \right. \\
 \left. , \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} p_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right\}. \tag{4.65}
 \end{aligned}$$

$\widehat{U}_1 = \{(u_1, v_1), p_1\}$ which imply the following expression

And we solve the second auxiliary problem from with boundary conditions

$$\begin{cases}
 \frac{\partial \widetilde{u}}{\partial y}|_{\partial\Omega_{hor}} = 0, & \widetilde{u}|_{\partial\Omega_{ver}} = 0 \\
 \widetilde{v}|_{\partial\Omega_{hor}} = 0, & \widetilde{v}|_{\partial\Omega_{ver}} = \widetilde{\alpha}
 \end{cases}$$

The solution to this auxiliary problem is of the form $\widetilde{U}_1 = \{(\widetilde{u}_1, \widetilde{v}_1), \widetilde{p}_1\}$ and this implies that

$$\widetilde{U}_1 = \left\{ \left(\sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{u}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{v}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right) \right\}$$

$$\left. , \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{p}_1(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right\}. \quad (4.66)$$

The above solution \hat{u}_1 and \tilde{u}_1 of auxiliary problem are added to obtain $U_1 = \hat{u}_1 + \tilde{u}_1$. Then

$$\begin{aligned} U_1 = & \left\{ \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} u_m(y) \sin m \left(x + \frac{\pi}{2} \right), \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} v_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right) \right. \\ & , \left. \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} p_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right\} \\ & + \left\{ \left(\sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{u}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{v}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right) \right. \\ & , \left. \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{p}_1(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right\}. \end{aligned}$$

From the above expression we obtain $U_1 = \hat{U}_1 + \tilde{U}_1$, given given as

$$U_1 = \left(\begin{array}{l} \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} u_m(y) \sin m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{u}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \\ \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} v_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{v}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \end{array} \right)$$

And for $p_1 = \hat{p}_1 + \tilde{p}_1$, we have

$$p_1 = \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} p_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{p}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right). \quad (4.67)$$

Then the 1st component of the first iteration is the exact solution of the problem

$$\left\{ \begin{array}{ll} -\Delta \hat{U}_{(1)} + \nabla \hat{P}_{(1)} = 0 & \text{in } \Omega \\ \nabla \cdot \hat{U}_{(1)} = 0 & \text{in } \Omega \\ U_{(1)}|_{\partial\Omega_{ver}} = 0, \quad U_{(1)}|_{\partial\Omega_{hor}} = \alpha + \beta_1 \\ \frac{\partial \hat{U}_{(1)}}{\partial x}|_{\partial\Omega_{ver}} & V_{(1)}|_{\partial\Omega_{hor}} = 0 \end{array} \right.$$

where β_1 is the trace of the solution \tilde{u}_1 to the second auxiliary problem at the 1st iteration.

The algorithm consists of the construction and solving of a sequence of pairs of auxiliary problems. The solution of each auxiliary problem is obtained through the use of the boundary conditions calculated from the trace of the solution of the previous auxiliary problem. However, the initial data for solving the first auxiliary problem is taken from the original problem. The sum of the solutions of the successive pair of auxiliary problems constitutes the solution of our initial problem. Similarly, the first term for the approximation of the second auxiliary problem is determined from the problem

$$\left\{ \begin{array}{ll} -\Delta \tilde{U}_{(1)} + \nabla \tilde{P}_{(1)} = 0 & \text{in } \Omega \\ \nabla \cdot \tilde{V}_{(1)} = 0 & \text{in } \Omega \\ \frac{\partial \tilde{U}_{(1)}}{\partial y}|_{\partial\Omega_{hor}} = 0, \quad \tilde{U}_{(1)}|_{\partial\Omega_{ver}} = 0 \\ \tilde{V}_{(1)}|_{\partial\Omega_{hor}} = 0, \quad \tilde{V}_{(1)}|_{\partial\Omega_{ver}} = \tilde{a}_1 \end{array} \right. \quad (4.68)$$

whose solutions are of the form

$$\tilde{U}_{(1)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right),$$

$$\tilde{V}_{(1)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right)$$

$$\tilde{P}_{(1)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right).$$

Thus the first iteration of the first auxiliary problem is given by

$$\hat{U}_2 = \left\{ \left(\hat{U}_{(1)}, \hat{V}_{(1)} \right), \hat{P}_{(1)} \right\}.$$

Then by substituting the iterated solutions of first auxiliary problem into above equation, we obtain

$$\begin{aligned} \hat{U}_2 = \left\{ \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} U_m(y) \sin m \left(x + \frac{\pi}{2} \right), \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} V_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right) \right. \\ \left. , \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} P_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right\}. \end{aligned} \quad (4.69)$$

And first iteration of second auxiliary problem is given by

$$\tilde{U}_2 = \left\{ \left(\tilde{U}_{(1)}, \tilde{V}_{(1)} \right), \tilde{P}_{(1)} \right\}.$$

Similarly by substituting iterated solutions of second auxiliary problem into \tilde{U}_2 , we obtain

$$\begin{aligned} \tilde{U}_2 = \left\{ \left(\sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right) \right. \\ \left. , \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right\}. \end{aligned} \quad (4.70)$$

By adding above equations, we get $U_{(2)} = \hat{U}_{(1)} + \tilde{U}_{(1)} + \hat{U}_{(2)} + \tilde{U}_{(2)}$, so that

$$\begin{aligned} U_2 = & \left\{ \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} U_m(y) \sin m \left(x + \frac{\pi}{2} \right), \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} V_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right) \right\} \\ & + \left\{ \left(\sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right) \right\} \\ & + \left\{ \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} U_m(y) \sin m \left(x + \frac{\pi}{2} \right), \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} V_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right) \right\}. \end{aligned}$$

$$+ \left\{ \left(\sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right) \right\} \quad (4.71)$$

From the above equation we get

$$\mathbf{U}_2 = 2 \left(\begin{array}{l} \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} U_m(y) \sin m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \\ \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} V_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \end{array} \right) \quad (4.72)$$

And for $P_2 = \hat{P}_{(1)} + \tilde{P}_{(1)} + \hat{P}_{(2)} + \tilde{P}_{(2)}$, we have

$$\begin{aligned} P_2 &= \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} P_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \\ &+ \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} P_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \\ P_2 &= 2 \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} P_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right). \end{aligned}$$

Then the 2^{nd} component of the first iteration is the exact solution of the problem

$$\left\{ \begin{array}{ll} -\Delta \hat{U}_{(2)} + \nabla \hat{P}_{(2)} = 0 & \text{in } \Omega \\ \nabla \cdot \hat{U}_{(2)} = 0 & \text{in } \Omega \\ U_{(2)}|_{\partial\Omega_{ver}} = 0, \quad U_{(2)}|_{\partial\Omega_{hor}} = \alpha + \beta_2 \\ \frac{\partial \hat{U}_{(2)}}{\partial x}|_{\partial\Omega_{ver}} = 0 & V_{(2)}|_{\partial\Omega_{hor}} = 0 \end{array} \right.$$

whose solution are of the form

$$\begin{aligned} \hat{U}_{(2)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \\ \hat{V}_{(2)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right), \\ \hat{P}_{(2)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right). \end{aligned}$$

where β_2 is the trace of the solution \tilde{u}_2 to the second auxiliary problem at the 2^{nd} iteration.

Similarly, the 2^{nd} term for the approximation of the second auxiliary problem is determined from the problem

$$\left\{ \begin{array}{ll} -\Delta \tilde{U}_{(2)} + \nabla \tilde{P}_{(2)} = 0 & \text{in } \Omega \\ \nabla \cdot \tilde{V}_{(2)} = 0 & \text{in } \Omega \\ \frac{\partial \tilde{U}_{(2)}}{\partial y}|_{\partial\Omega_{hor}} = 0, \quad \tilde{U}_{(2)}|_{\partial\Omega_{ver}} = 0 \\ \tilde{V}_{(2)}|_{\partial\Omega_{hor}} = 0, \quad \tilde{V}_{(2)}|_{\partial\Omega_{ver}} = \tilde{a}_2 \end{array} \right. \quad (4.73)$$

whose solutions are of the form

$$\tilde{U}_{(2)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right),$$

$$\tilde{V}_{(2)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right),$$

$$\tilde{P}_{(2)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right).$$

Thus the 2^{nd} iteration of the first auxiliary problem is given by

$$\hat{U}_3 = \left\{ \left(\hat{U}_{(2)}, \hat{V}_{(2)} \right), \hat{P}_{(2)} \right\}.$$

Then by substituting the iterated solutions of first auxiliary problem into above equation, we obtain

$$\begin{aligned} \hat{U}_3 = & \left\{ \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} U_m(y) \sin m \left(x + \frac{\pi}{2} \right), \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} V_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right) \right. \\ & \left. , \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} P_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right\}. \end{aligned} \quad (4.74)$$

And first iteration of second auxiliary problem is given by

$$\tilde{U}_3 = \left\{ \left(\tilde{U}_{(2)}, \tilde{V}_{(2)} \right), \tilde{P}_{(2)} \right\}$$

Similarly by substituting iterated solutions of second auxiliary problem into \tilde{U}_2 , we obtain

$$\begin{aligned} \tilde{U}_3 = & \left\{ \left(\sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right) \right. \\ & \left. , \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right\}. \end{aligned} \quad (4.75)$$

By adding $U_3 = \hat{U}_{(1)} + \tilde{U}_{(1)} + \hat{U}_{(2)} + \tilde{U}_{(2)} + \hat{U}_{(3)} + \tilde{U}_{(3)}$

$$U_3 = 3 \left(\begin{array}{l} \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} U_m(y) \sin m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \\ \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} V_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \end{array} \right) \quad (4.76)$$

$$P_3 = \hat{P}_{(1)} + \tilde{P}_{(1)} + \hat{P}_{(2)} + \tilde{P}_{(2)} + \hat{P}_{(3)} + \tilde{P}_{(3)}$$

$$P_3 = 3 \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} P_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right).$$

Then the 3^{rd} component of the first iteration is the exact solution of the problem

$$\left\{ \begin{array}{ll} -\Delta \widehat{U}_{(3)} + \nabla \widehat{P}_{(3)} = 0 & \text{in } \Omega \\ \nabla \cdot \widehat{U}_{(3)} = 0 & \text{in } \Omega \\ U_{(3)}|_{\partial\Omega_{ver}} = 0, \quad U_{(3)}|_{\partial\Omega_{hor}} = \alpha + \beta_3 & \\ \frac{\partial \widehat{U}_{(3)}}{\partial x}|_{\partial\Omega_{ver}} = 0 & \quad V_{(3)}|_{\partial\Omega_{hor}} = 0 \end{array} \right.$$

whose solution are of the form

$$\begin{aligned} \widehat{U}_{(3)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \\ \widehat{V}_{(3)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right), \\ \widehat{P}_{(3)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right). \end{aligned}$$

where β_3 is the trace of the solution \widetilde{U}_3 to the second auxiliary problem at the 3^{rd} iteration.

Similarly, the 3^{rd} term for the approximation of the second auxiliary problem is determined from the problem

$$\left\{ \begin{array}{ll} -\Delta \widetilde{U}_{(3)} + \nabla \widetilde{P}_{(3)} = 0 & \text{in } \Omega \\ \nabla \cdot \widetilde{V}_{(3)} = 0 & \text{in } \Omega \\ \frac{\partial \widetilde{U}_{(3)}}{\partial y}|_{\partial\Omega_{hor}} = 0, \quad \widetilde{U}_{(3)}|_{\partial\Omega_{ver}} = 0 & \\ \widetilde{V}_{(3)}|_{\partial\Omega_{hor}} = 0, \quad \widetilde{V}_{(3)}|_{\partial\Omega_{ver}} = \widetilde{a}_2 & \end{array} \right. \quad (4.77)$$

whose solutions are of the form

$$\begin{aligned} \widetilde{U}_{(3)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \\ \widetilde{V}_{(3)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right), \\ \widetilde{P}_{(3)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right). \end{aligned}$$

Thus the 3^{rd} iteration of the first auxiliary problem is given by

$$\widehat{U}_4 = \left\{ \left(\widehat{U}_{(3)}, \widehat{V}_{(3)} \right), \widehat{P}_{(3)} \right\}.$$

Then by substituting the iterated solutions of first auxiliary problem into above equation, we obtain

$$\begin{aligned} \widehat{U}_4 &= \left\{ \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} U_m(y) \sin m \left(x + \frac{\pi}{2} \right), \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} V_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right) \right. \\ &\quad \left. , \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} P_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right\}. \end{aligned} \quad (4.78)$$

And first iteration of second auxiliary problem is given by

$$\tilde{U}_4 = \left\{ \left(\tilde{U}_{(3)}, \tilde{V}_{(3)} \right), \tilde{P}_{(3)} \right\}.$$

Similarly by substituting iterated solutions of second auxiliary problem into \tilde{U}_3 , we obtain

$$\begin{aligned} \tilde{U}_4 = \left\{ \left(\sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right) \right. \\ \left. , \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right\} \end{aligned} \quad (4.79)$$

By adding $U_4 = \hat{U}_{(1)} + \tilde{U}_{(1)} + \hat{U}_{(2)} + \tilde{U}_{(2)} + \hat{U}_{(3)} + \tilde{U}_{(3)} + \hat{U}_{(4)} + \tilde{U}_{(4)}$

$$\begin{aligned} U_4 = 4 \left(\begin{array}{l} \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} U_m(y) \sin m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \\ \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} V_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \end{array} \right) \\ P_4 = \hat{P}_{(1)} + \tilde{P}_{(1)} + \hat{P}_{(2)} + \tilde{P}_{(2)} + \hat{P}_{(3)} + \tilde{P}_{(3)} + \hat{P}_{(4)} + \tilde{P}_{(4)}. \\ P_4 = 4 \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} P_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right). \end{aligned}$$

Then the 4th component of the first iteration is the exact solution of the problem

$$\left\{ \begin{array}{ll} -\Delta \hat{U}_{(4)} + \nabla \hat{P}_{(4)} = 0 & \text{in } \Omega \\ \nabla \cdot \hat{U}_{(4)} = 0 & \text{in } \Omega \\ U_{(4)}|_{\partial\Omega_{ver}} = 0, \quad U_{(4)}|_{\partial\Omega_{hor}} = \alpha + \beta_4 & \\ \frac{\partial \hat{U}_{(4)}}{\partial x}|_{\partial\Omega_{ver}} = 0 & V_{(4)}|_{\partial\Omega_{hor}} = 0 \end{array} \right.$$

whose solution are of the form

$$\begin{aligned} \hat{U}_{(4)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \\ \hat{V}_{(4)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right), \\ \hat{P}_{(4)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right). \end{aligned}$$

where β_4 is the trace of the solution \tilde{U}_4 to the second auxiliary problem at the 4th iteration.

Similarly, the 4th term for the approximation of the second auxiliary problem is determined from the problem

$$\left\{ \begin{array}{l} -\Delta \tilde{U}_{(4)} + \nabla \tilde{P}_{(4)} = 0 \quad \text{in } \Omega \\ \nabla \cdot \tilde{V}_{(4)} = 0 \quad \text{in } \Omega \\ \frac{\partial \tilde{U}_{(4)}}{\partial y} \Big|_{\partial \Omega_{hor}} = 0, \quad \tilde{U}_{(4)} \Big|_{\partial \Omega_{ver}} = 0 \\ \tilde{V}_{(4)} \Big|_{\partial \Omega_{hor}} = 0, \quad \tilde{V}_{(4)} \Big|_{\partial \Omega_{ver}} = \tilde{a}_2 \end{array} \right. \quad (4.80)$$

whose solutions are of the form

$$\tilde{U}_{(4)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right),$$

$$\tilde{V}_{(4)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right),$$

$$\tilde{P}_{(4)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right).$$

Thus the 4th iteration of the first auxiliary problem is given by

$$\hat{U}_5 = \left\{ \left(\hat{U}_{(4)}, \hat{V}_{(4)} \right), \hat{P}_{(4)} \right\}.$$

Then by substituting the iterated solutions of first auxiliary problem into above equation, we obtain

$$\hat{U}_5 = \left\{ \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} U_m(y) \sin m \left(x + \frac{\pi}{2} \right), \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} V_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right), \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} P_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right\}. \quad (4.81)$$

And first iteration of second auxiliary problem is given by

$$\tilde{U}_5 = \left\{ \left(\tilde{U}_{(4)}, \tilde{V}_{(4)} \right), \tilde{P}_{(4)} \right\}.$$

Similarly by substituting iterated solutions of second auxiliary problem into \tilde{U}_5 , we obtain

$$\tilde{U}_5 = \left\{ \left(\sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right), \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right\}. \quad (4.82)$$

By adding $U_5 = \hat{U}_{(1)} + \tilde{U}_{(1)} + \hat{U}_{(2)} + \tilde{U}_{(2)} + \hat{U}_{(3)} + \tilde{U}_{(3)} + \hat{U}_{(4)} + \tilde{U}_{(4)} + \hat{U}_{(5)} + \tilde{U}_{(5)}$

$$U_5 = 5 \left(\begin{array}{l} \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} U_m(y) \sin m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \\ \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} V_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \end{array} \right) \quad (4.83)$$

$$P_5 = \widehat{P}_{(1)} + \widetilde{P}_{(1)} + \widehat{P}_{(2)} + \widetilde{P}_{(2)} + \widehat{P}_{(3)} + \widetilde{P}_{(3)} + \widehat{P}_{(4)} + \widetilde{P}_{(4)} + \widehat{P}_{(5)} + \widetilde{P}_{(5)}$$

$$P_5 = 5 \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} P_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right).$$

Then the 5th component of the first iteration is the exact solution of the problem

$$\left\{ \begin{array}{ll} -\Delta \widehat{U}_{(5)} + \nabla \widehat{P}_{(5)} = 0 & \text{in } \Omega \\ \nabla \cdot \widehat{U}_{(5)} = 0 & \text{in } \Omega \\ U_{(5)}|_{\partial\Omega_{ver}} = 0, \quad U_{(5)}|_{\partial\Omega_{hor}} = \alpha + \beta_5 \\ \frac{\partial \widehat{U}_{(5)}}{\partial x}|_{\partial\Omega_{ver}} = 0 & V_{(5)}|_{\partial\Omega_{hor}} = 0 \end{array} \right.$$

whose solution are of the form

$$\widehat{U}_{(5)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right),$$

$$\widehat{V}_{(5)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right),$$

$$\widehat{P}_{(5)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right).$$

where β_5 is the trace of the solution \widetilde{U}_5 to the second auxiliary problem at the 5th iteration.

Similarly, the 5th term for the approximation of the second auxiliary problem is determined from the problem

$$\left\{ \begin{array}{ll} -\Delta \widetilde{U}_{(5)} + \nabla \widetilde{P}_{(5)} = 0 & \text{in } \Omega \\ \nabla \cdot \widetilde{V}_{(5)} = 0 & \text{in } \Omega \\ \frac{\partial \widetilde{U}_{(5)}}{\partial y}|_{\partial\Omega_{hor}} = 0, \quad \widetilde{U}_{(5)}|_{\partial\Omega_{ver}} = 0 \\ \widetilde{V}_{(5)}|_{\partial\Omega_{hor}} = 0, \quad \widetilde{V}_{(5)}|_{\partial\Omega_{ver}} = \widetilde{a}_5 \end{array} \right. \quad (4.84)$$

whose solutions are of the form

$$\widetilde{U}_{(5)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right),$$

$$\widetilde{V}_{(5)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right),$$

$$\widetilde{P}_{(5)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right).$$

Thus the 6th iteration of the first auxiliary problem is given by

$$\widehat{U}_6 = \left\{ \left(\widehat{U}_{(5)}, \widehat{V}_{(5)} \right), \widehat{P}_{(5)} \right\}.$$

Then by substituting the iterated solutions of first auxiliary problem into above equation, we obtain

$$\hat{U}_6 = \left\{ \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} U_m(y) \sin m \left(x + \frac{\pi}{2} \right), \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} V_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right), \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} P_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right\}. \quad (4.85)$$

And first iteration of second auxiliary problem is given by

$$\tilde{U}_6 = \left\{ \left(\tilde{U}_{(5)}, \tilde{V}_{(5)} \right), \tilde{P}_{(5)} \right\}.$$

Similarly by substituting iterated solutions of second auxiliary problem into \tilde{U}_6 , we obtain

$$\tilde{U}_6 = \left\{ \left(\sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right), \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right\}. \quad (4.86)$$

By adding $U_6 = \hat{U}_{(1)} + \tilde{U}_{(1)} + \hat{U}_{(2)} + \tilde{U}_{(2)} + \hat{U}_{(3)} + \tilde{U}_{(3)} + \hat{U}_{(4)} + \tilde{U}_{(4)} + \hat{U}_{(5)} + \tilde{U}_{(5)} + \hat{U}_{(6)} + \tilde{U}_{(6)}$

$$U_6 = 6 \left(\begin{array}{l} \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} U_m(y) \sin m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \\ \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} V_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \end{array} \right) \quad (4.87)$$

$$P_6 = \hat{P}_{(1)} + \tilde{P}_{(1)} + \hat{P}_{(2)} + \tilde{P}_{(2)} + \hat{P}_{(3)} + \tilde{P}_{(3)} + \hat{P}_{(4)} + \tilde{P}_{(4)} + \hat{P}_{(5)} + \tilde{P}_{(5)} + \hat{P}_{(6)} + \tilde{P}_{(6)}.$$

$$P_6 = 6 \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} P_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right).$$

Then the 6th component of the first iteration is the exact solution of the problem

$$\left\{ \begin{array}{ll} -\Delta \hat{U}_{(6)} + \nabla \hat{P}_{(6)} = 0 & \text{in } \Omega \\ \nabla \cdot \hat{U}_{(6)} = 0 & \text{in } \Omega \\ U_{(6)}|_{\partial\Omega_{ver}} = 0, & U_{(6)}|_{\partial\Omega_{hor}} = \alpha + \beta_6 \\ \frac{\partial \hat{U}_{(6)}}{\partial x}|_{\partial\Omega_{ver}} = 0 & V_{(6)}|_{\partial\Omega_{hor}} = 0 \end{array} \right.$$

whose solution are of the form

$$\hat{U}_{(6)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right),$$

$$\hat{V}_{(6)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right),$$

$$\hat{P}_{(6)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right).$$

where β_6 is the trace of the solution \tilde{U}_6 to the second auxiliary problem at the 6th iteration.

Similarly, the 6th term for the approximation of the second auxiliary problem is determined from the problem

$$\begin{cases} -\Delta \tilde{U}_{(6)} + \nabla \tilde{P}_{(6)} = 0 & \text{in } \Omega \\ \nabla \cdot \tilde{V}_{(6)} = 0 & \text{in } \Omega \\ \frac{\partial \tilde{U}_{(6)}}{\partial y} |_{\partial\Omega_{hor}} = 0, \quad \tilde{U}_{(6)} |_{\partial\Omega_{ver}} = 0 \\ \tilde{V}_{(6)} |_{\partial\Omega_{hor}} = 0, \quad \tilde{V}_{(6)} |_{\partial\Omega_{ver}} = \tilde{a}_2 \end{cases} \quad (4.88)$$

whose solutions are of the form

$$\tilde{U}_{(6)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right),$$

$$\tilde{V}_{(6)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right),$$

$$\tilde{P}_{(6)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right).$$

Thus the 6th iteration of the first auxiliary problem is given by

$$\hat{U}_7 = \left\{ \left(\hat{U}_{(7)}, \hat{V}_7 \right), \hat{P}_{(7)} \right\}.$$

Then by substituting the iterated solutions of first auxiliary problem into above equation, we obtain

$$\begin{aligned} \hat{U}_7 = & \left\{ \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} U_m(y) \sin m \left(x + \frac{\pi}{2} \right), \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} V_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right) \right. \\ & \left. , \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} P_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right\}. \end{aligned} \quad (4.89)$$

And first iteration of second auxiliary problem is given by

$$\tilde{U}_7 = \left\{ \left(\tilde{U}_{(6)}, \tilde{V}_{(6)} \right), \tilde{P}_{(6)} \right\}.$$

Similarly by substituting iterated solutions of second auxiliary problem into \tilde{U}_7 , we obtain

$$\begin{aligned} \tilde{U}_7 = & \left\{ \left(\sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right) \right. \\ & \left. , \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right\}. \end{aligned} \quad (4.90)$$

By adding

$$U_7 = \widehat{U}_{(1)} + \widetilde{U}_{(1)} + \widehat{U}_{(2)} + \widetilde{U}_{(2)} + \widehat{U}_{(3)} + \widetilde{U}_{(3)} + \widehat{U}_{(4)} + \widetilde{U}_{(4)} + \widehat{U}_{(5)} + \widetilde{U}_{(5)} + \widehat{U}_{(6)} + \widetilde{U}_{(6)} + \widehat{U}_{(7)} + \widetilde{U}_{(7)}. \quad (4.91)$$

$$U_7 = 7 \left(\begin{array}{l} \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} U_m(y) \sin m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \\ \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} V_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \end{array} \right). \quad (4.92)$$

$$P_7 = \widehat{P}_{(1)} + \widetilde{P}_{(1)} + \widehat{P}_{(2)} + \widetilde{P}_{(2)} + \widehat{P}_{(3)} + \widetilde{P}_{(3)} + \widehat{P}_{(4)} + \widetilde{P}_{(4)} + \widehat{P}_{(5)} + \widetilde{P}_{(5)} + \widehat{P}_{(6)} + \widetilde{P}_{(6)} + \widehat{P}_{(7)}.$$

$$P_7 = 7 \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} P_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right).$$

Then the 7^{th} component of the first iteration is the exact solution of the problem

$$\left\{ \begin{array}{ll} -\Delta \widehat{U}_{(7)} + \nabla \widehat{P}_{(7)} = 0 & \text{in } \Omega \\ \nabla \cdot \widehat{U}_{(7)} = 0 & \text{in } \Omega \\ U_{(7)}|_{\partial\Omega_{ver}} = 0, \quad U_{(7)}|_{\partial\Omega_{hor}} = \alpha + \beta_7 & \\ \frac{\partial \widehat{U}_{(7)}}{\partial x}|_{\partial\Omega_{ver}} = 0 & V_{(7)}|_{\partial\Omega_{hor}} = 0 \end{array} \right.$$

whose solution are of the form

$$\widehat{U}_{(7)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right),$$

$$\widehat{V}_{(7)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right),$$

$$\widehat{P}_{(7)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right).$$

where β_7 is the trace of the solution \widetilde{U}_7 to the second auxiliary problem at the 7^{th} iteration.

Similarly, the 7^{th} term for the approximation of the second auxiliary problem is determined from the problem

$$\left\{ \begin{array}{ll} -\Delta \widetilde{U}_{(7)} + \nabla \widetilde{P}_{(7)} = 0 & \text{in } \Omega \\ \nabla \cdot \widetilde{V}_{(7)} = 0 & \text{in } \Omega \\ \frac{\partial \widetilde{U}_{(7)}}{\partial y}|_{\partial\Omega_{hor}} = 0, \quad \widetilde{U}_{(7)}|_{\partial\Omega_{ver}} = 0 & \\ \widetilde{V}_{(7)}|_{\partial\Omega_{hor}} = 0, \quad \widetilde{V}_{(7)}|_{\partial\Omega_{ver}} = \widetilde{a}_2 & \end{array} \right. \quad (4.93)$$

whose solutions are of the form

$$\tilde{U}_{(\tau)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right),$$

$$\tilde{V}_{(\tau)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right),$$

$$\tilde{P}_{(\tau)} = \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right).$$

Thus the 7th iteration of the first auxiliary problem is given by

$$\hat{U}_8 = \left\{ \left(\hat{U}_{(\tau)}, \hat{V}_{(\tau)} \right), \hat{P}_{(\tau)} \right\}$$

Then by substituting the iterated solutions of first auxiliary problem into above equation, we obtain

$$\begin{aligned} \hat{U}_8 = \left\{ \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} U_m(y) \sin m \left(x + \frac{\pi}{2} \right), \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} V_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right) \right. \\ \left. , \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} P_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right\}. \end{aligned} \quad (4.94)$$

And first iteration of second auxiliary problem is given by

$$\tilde{U}_8 = \left\{ \left(\tilde{U}_{(\tau)}, \tilde{V}_{(\tau)} \right), \tilde{P}_{(\tau)} \right\}.$$

Similarly by substituting iterated solutions of second auxiliary problem into \tilde{U}_8 , we obtain

$$\begin{aligned} \tilde{U}_8 = \left\{ \left(\sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right) \right. \\ \left. , \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right\}. \end{aligned} \quad (4.95)$$

By adding

$$\begin{aligned} U_8 = \hat{U}_{(1)} + \tilde{U}_{(1)} + \hat{U}_{(2)} + \tilde{U}_{(2)} + \hat{U}_{(3)} + \tilde{U}_{(3)} + \hat{U}_{(4)} + \tilde{U}_{(4)} + \hat{U}_{(5)} + \tilde{U}_{(5)} + \hat{U}_{(6)} + \tilde{U}_{(6)} + \hat{U}_{(7)} \\ + \tilde{U}_{(7)} + \hat{U}_{(8)} + \tilde{U}_{(8)}. \end{aligned} \quad (4.96)$$

$$U_8 = 8 \left(\begin{array}{l} \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} U_m(y) \sin m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \\ \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} V_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \end{array} \right)$$

$$\begin{aligned}
 P_8 = & \widehat{P}_{(1)} + \widetilde{P}_{(1)} + \widehat{P}_{(2)} + \widetilde{P}_{(2)} + \widehat{P}_{(3)} + \widetilde{P}_{(3)} + \widehat{P}_{(4)} + \widetilde{P}_{(4)} + \widehat{P}_{(5)} + \widetilde{P}_{(5)} + \widehat{P}_{(6)} \\
 & + \widetilde{P}_{(6)} + \widehat{P}_{(7)} + \widetilde{P}_{(7)} + \widehat{P}_{(8)} + \widetilde{P}_{(8)}.
 \end{aligned} \tag{4.97}$$

$$P_8 = 8 \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} P_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right).$$

Then the 8^{th} component of the first iteration is the exact solution of the problem

$$\left\{ \begin{array}{ll}
 -\Delta \widehat{U}_{(8)} + \nabla \widehat{P}_{(8)} = 0 & \text{in } \Omega \\
 \nabla \cdot \widehat{U}_{(8)} = 0 & \text{in } \Omega \\
 U_{(8)}|_{\partial\Omega_{ver}} = 0, & U_{(8)}|_{\partial\Omega_{hor}} = \alpha + \beta_8 \\
 \frac{\partial \widehat{U}_{(8)}}{\partial x}|_{\partial\Omega_{ver}} = 0 & V_{(8)}|_{\partial\Omega_{hor}} = 0
 \end{array} \right.$$

whose solution are of the form

$$\begin{aligned}
 \widehat{U}_{(8)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \\
 \widehat{V}_{(8)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right), \\
 \widehat{P}_{(8)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right).
 \end{aligned}$$

where β_8 is the trace of the solution \widehat{U}_8 to the second auxiliary problem at the 8^{th} iteration.

Similarly, the 8^{th} term for the approximation of the second auxiliary problem is determined from the problem

$$\left\{ \begin{array}{ll}
 -\Delta \widetilde{U}_{(8)} + \nabla \widetilde{P}_{(8)} = 0 & \text{in } \Omega \\
 \nabla \cdot \widetilde{V}_{(8)} = 0 & \text{in } \Omega \\
 \frac{\partial \widetilde{U}_{(8)}}{\partial y}|_{\partial\Omega_{hor}} = 0, & \widetilde{U}_{(8)}|_{\partial\Omega_{ver}} = 0 \\
 \widetilde{V}_{(8)}|_{\partial\Omega_{hor}} = 0, & \widetilde{V}_{(8)}|_{\partial\Omega_{ver}} = \widetilde{a}_8
 \end{array} \right. \tag{4.98}$$

whose solutions are of the form

$$\begin{aligned}
 \widetilde{U}_{(8)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \\
 \widetilde{V}_{(8)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right), \\
 \widetilde{P}_{(8)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right).
 \end{aligned}$$

Thus the 8^{th} iteration of the first auxiliary problem is given by

$$\widehat{U}_9 = \left\{ \left(\widehat{U}_{(8)}, \widehat{V}_{(8)} \right), \widehat{P}_{(8)} \right\}.$$

Then by substituting the iterated solutions of first auxiliary problem into above equation, we obtain

$$\hat{U}_9 = \left\{ \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} U_m(y) \sin m \left(x + \frac{\pi}{2} \right), \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} V_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right), \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} P_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right\}. \quad (4.99)$$

And first iteration of second auxiliary problem is given by

$$\tilde{U}_9 = \left\{ \left(\tilde{U}_{(8)}, \tilde{V}_{(8)} \right), \tilde{P}_{(8)} \right\}.$$

Similarly by substituting iterated solutions of second auxiliary problem into \tilde{U}_9 , we obtain

$$\tilde{U}_9 = \left\{ \left(\sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right), \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right\}. \quad (4.100)$$

By adding

$$U_9 = \hat{U}_{(1)} + \tilde{U}_{(1)} + \hat{U}_{(2)} + \tilde{U}_{(2)} + \hat{U}_{(3)} + \tilde{U}_{(3)} + \hat{U}_{(4)} + \tilde{U}_{(4)} + \hat{U}_{(5)} + \tilde{U}_{(5)} + \hat{U}_{(6)} + \tilde{U}_{(6)} + \hat{U}_{(7)} + \tilde{U}_{(7)} + \hat{U}_{(8)} + \tilde{U}_{(8)} + \hat{U}_{(9)} + \tilde{U}_{(9)}. \quad (4.101)$$

$$U_9 = 9 \left(\begin{array}{l} \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} U_m(y) \sin m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \\ \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} V_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \end{array} \right)$$

$$P_9 = \hat{P}_{(1)} + \tilde{P}_{(1)} + \hat{P}_{(2)} + \tilde{P}_{(2)} + \hat{P}_{(3)} + \tilde{P}_{(3)} + \hat{P}_{(4)} + \tilde{P}_{(4)} + \hat{P}_{(5)} + \tilde{P}_{(5)} + \hat{P}_{(6)} + \tilde{P}_{(6)} + \hat{P}_{(7)} + \tilde{P}_{(7)} + \hat{P}_{(8)} + \tilde{P}_{(8)} + \hat{P}_{(9)} + \tilde{P}_{(9)}. \quad (4.102)$$

$$P_9 = 9 \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} P_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right)$$

Then the 9^{th} component of the first iteration is the exact solution of the problem

$$\left\{ \begin{array}{ll} -\Delta \hat{U}_{(9)} + \nabla \hat{P}_{(9)} = 0 & \text{in } \Omega \\ \nabla \cdot \hat{U}_{(9)} = 0 & \text{in } \Omega \\ U_{(9)}|_{\partial\Omega_{ver}} = 0, \quad U_{(9)}|_{\partial\Omega_{hor}} = \alpha + \beta_9 \\ \frac{\partial \hat{U}_{(9)}}{\partial x}|_{\partial\Omega_{ver}} = 0 & V_{(9)}|_{\partial\Omega_{hor}} = 0 \end{array} \right.$$

whose solution are of the form

$$\begin{aligned}\widehat{U}_{(9)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \\ \widehat{V}_{(9)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right), \\ \widehat{P}_{(9)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right).\end{aligned}$$

where β_9 is the trace of the solution \widetilde{U}_9 to the second auxiliary problem at the 9^{th} iteration.

Similarly, the 9^{th} term for the approximation of the second auxiliary problem is determined from the problem

$$\left\{ \begin{array}{ll} -\Delta \widetilde{U}_{(9)} + \nabla \widetilde{P}_{(9)} = 0 & \text{in } \Omega \\ \nabla \cdot \widetilde{V}_{(9)} = 0 & \text{in } \Omega \\ \frac{\partial \widetilde{U}_{(9)}}{\partial y} \Big|_{\partial\Omega_{hor}} = 0, \quad \widetilde{U}_{(9)} \Big|_{\partial\Omega_{ver}} = 0, \quad \widetilde{V}_{(9)} \Big|_{\partial\Omega_{hor}} = 0, \quad \widetilde{V}_{(9)} \Big|_{\partial\Omega_{ver}} = \widetilde{a}_9 \end{array} \right. \quad (4.103)$$

whose solutions are of the form

$$\begin{aligned}\widetilde{U}_{(9)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \\ \widetilde{V}_{(9)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right), \\ \widetilde{P}_{(9)} &= \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \widetilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right).\end{aligned}$$

Thus the 10^{th} iteration of the first auxiliary problem is given by

$$\widehat{U}_{10} = \left\{ \left(\widehat{U}_{(9)}, \widehat{V}_{(9)} \right), \widehat{P}_{(9)} \right\}.$$

Then by substituting the iterated solutions of first auxiliary problem into above equation, we obtain

$$\begin{aligned}\widehat{U}_9 &= \left\{ \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} U_m(y) \sin m \left(x + \frac{\pi}{2} \right), \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} V_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right) \right. \\ &\quad \left. , \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} P_m(y) \cos m \left(x + \frac{\pi}{2} \right) \right\}. \end{aligned} \quad (4.104)$$

And first iteration of second auxiliary problem is given by

$$\widetilde{U}_{10} = \left\{ \left(\widetilde{U}_{(9)}, \widetilde{V}_{(9)} \right), \widetilde{P}_{(9)} \right\}.$$

Similarly by substituting iterated solutions of second auxiliary problem into \tilde{U}_9 , we obtain

$$\tilde{U}_{10} = \left\{ \left(\sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right), \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right), \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right\}. \quad (4.105)$$

By adding

$$U_{10} = \hat{U}_{(1)} + \tilde{U}_{(1)} + \hat{U}_{(2)} + \tilde{U}_{(2)} + \hat{U}_{(3)} + \tilde{U}_{(3)} + \hat{U}_{(4)} + \tilde{U}_{(4)} + \hat{U}_{(5)} + \tilde{U}_{(5)} + \hat{U}_{(6)} + \tilde{U}_{(6)} + \hat{U}_{(7)} + \tilde{U}_{(7)} + \hat{U}_{(8)} + \tilde{U}_{(8)} + \hat{U}_{(9)} + \tilde{U}_{(9)} + \hat{U}_{(10)} + \tilde{U}_{(10)} \quad (4.106)$$

$$U_{10} = 10 \left(\begin{array}{l} \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} U_m(y) \sin m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{U}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \\ \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} V_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{V}_m(x) \sin m \left(\frac{y}{d} + \frac{\pi}{2} \right) \end{array} \right) \quad (4.107)$$

$$P_{10} = \hat{P}_{(1)} + \tilde{P}_{(1)} + \hat{P}_{(2)} + \tilde{P}_{(2)} + \hat{P}_{(3)} + \tilde{P}_{(3)} + \hat{P}_{(4)} + \tilde{P}_{(4)} + \hat{P}_{(5)} + \tilde{P}_{(5)} + \hat{P}_{(6)} + \tilde{P}_{(6)} + \hat{P}_{(7)} + \tilde{P}_{(7)} + \hat{P}_{(8)} + \tilde{P}_{(8)} + \hat{P}_{(9)} + \tilde{P}_{(9)} + \hat{P}_{(10)} + \tilde{P}_{(10)} \quad (4.108)$$

$$P_{10} = 10 \left(\sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} P_m(y) \cos m \left(x + \frac{\pi}{2} \right) + \sqrt{\frac{2}{d\pi}} \sum_{m=1}^{\infty} \tilde{P}_m(x) \cos m \left(\frac{y}{d} + \frac{\pi}{2} \right) \right). \quad (4.109)$$

For the successive iteration, the nonzero boundary condition at the first step is taken equal to

$$u|_{\partial\Omega_{hor}} = \alpha_1$$

The procedure is then cyclically repeated n times, which is the number of iteration performed. The algorithm generates the approximation

$$U_{(n)} = \sum_{i=1}^n (\hat{U}_i + \tilde{U}_i). \quad (4.110)$$

and

$$P_{(n)} = \sum_{i=1}^n (\hat{P}_i + \tilde{P}_i). \quad (4.111)$$

The pairs $\{U_n, P_n\}$ and $\{\tilde{U}_n, \tilde{P}_n\}$ are the solutions to the corresponding auxiliary problems at the n^{th} iteration

$$\left\{ \begin{array}{ll} -\Delta U + \nabla P_{(n)} = 0 & in \ \Omega \\ \nabla \cdot \hat{U}_{(n)} = 0 & in \ \Omega \\ U_{(n)}|_{\partial\Omega_{ver}} = 0, V_{(n)}|_{\partial\Omega} = 0, U_{(n)}|_{\partial\Omega_{hor}} = \alpha + \beta_n \end{array} \right.$$

Chapter 5

Numerical Method for Solving Stokes problem

In this chapter, we discuss the full implementation of the numerical method on rectangular and irregular domains. We use Vorticity stream function approach to solve Stokes problem on a rectangular domain and explicit second order central difference scheme for an irregular domain.

5.1 Numerical Solution using vorticity Stream Function approach on rectangular domain

Below is the expanded Stoke's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x} = 0, \quad (5.1)$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - \frac{\partial p}{\partial y} = 0, \quad (5.2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (5.3)$$

We partially derivate 5.1 with respect to y and 5.2 with respect to x to obtain the following

$$\frac{\partial}{\partial y} \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} \right) - \frac{\partial^2 p}{\partial y \partial x} = 0. \quad (5.4)$$

$$\frac{\partial}{\partial x} \left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial y^2} \right) - \frac{\partial^2 p}{\partial x \partial y} = 0. \quad (5.5)$$

By assuming that pressure function is piece-wise continuous function within the above defined intervals, implies that

$$\frac{\partial^2 p}{\partial y \partial x} = \frac{\partial^2 p}{\partial x \partial y}. \quad (5.6)$$

By substituting 5.2 into 5.1 we obtain the following

$$\frac{\partial}{\partial x} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \frac{\partial}{\partial y} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0. \quad (5.7)$$

From continuity equation we have the following

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (5.8)$$

this imply that

$$\frac{\partial^2 u}{\partial x^2} = - \frac{\partial^2 v}{\partial x \partial y}. \quad (5.9)$$

and

$$\frac{\partial^2 v}{\partial x^2} = - \frac{\partial^2 u}{\partial y \partial x} \quad (5.10)$$

By using the above expression into 5.1 and 5.2 we obtain the following

$$\frac{\partial}{\partial x} \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial y \partial x} \right) - \frac{\partial}{\partial y} \left(-\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} \right) = 0, \quad (5.11)$$

$$\frac{\partial^2}{\partial x^2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = 0, \quad (5.12)$$

let

$$W = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \quad (5.13)$$

having boundary conditions

$$u|_{\partial\Omega_{ver}} = 0, u|_{\partial\Omega_{hor}=a}, \quad (5.14)$$

$$\frac{\partial v}{\partial x}|_{\partial\Omega_{ver}} = 0, v|_{\partial\Omega_{hor}} = 0. \quad (5.15)$$

From the boundary conditions given above we can have boundary conditions for vorticity function,thus

$$W|_{\partial\Omega_{ver}} = 0 - 0 = 0, \quad (5.16)$$

$$W|_{\partial\Omega_{hor}} = 0 - \frac{\partial u}{\partial y}|_{\partial\Omega_{hor}} = 0. \quad (5.17)$$

By discretizing above equation we have the following

$$\frac{W_{i+1,j} - 2W_{i,j} + W_{i-1,j}}{h_x^2} + \frac{W_{i,j+1} - 2W_{i,j} + W_{i,j-1}}{h_y^2} = 0.$$

for $h_x = h_y = h$ imply that

$$W_{i,j} = \frac{1}{4} [W_{i+1,j} + W_{i-1,j} + W_{i,j+1} + W_{i,j-1}]. \quad (5.18)$$

5.2 Numerical solution of stokes problem on irregular domain

Here we consider irregular interior grid points and points on the boundary where Cauchy or Neumann conditions are supposed to be valid. We deal with this situation through the construction of appropriate difference equations of given differential operators. Our task is to approximate the differential operators at grid points that are lying near to the curved part of the boundary. The approximations are linear combinations of the values $u_{i-1,j}, u_{i,j}$ and $u_{i+1,j}$ and of $u_{i,j-1}, u_{i,j}$ and $u_{i,j+1}$ respectively. Assume that $u(x, y), v(x, y)$ and $p(x, y)$ are continuous and differentiable function, thus by using Taylor series expansion, the following relationship hold for function values of the corresponding points.

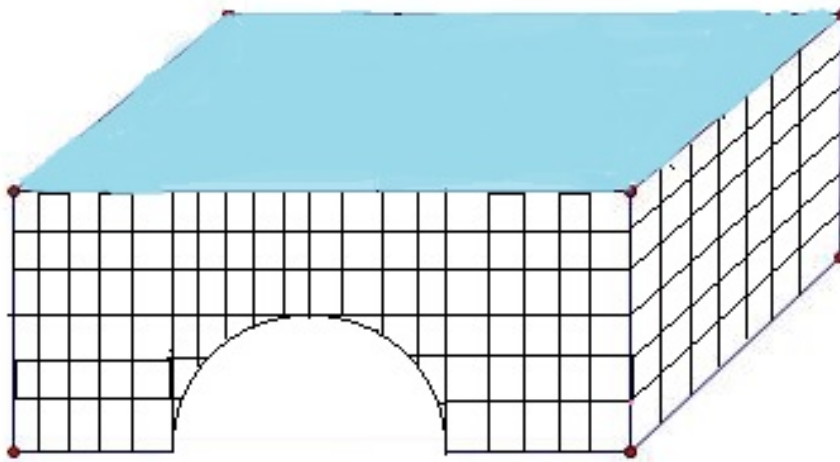


Figure 5.1: Irregular domain in 3D

The positions of irregular points 29,30,36,41,45,52,56,60,65 are indicated below. The points lie on the domain $x = -\frac{\pi}{2}$ to $\frac{\pi}{2}$ and $y = -\frac{d\pi}{2}$ to $\frac{d\pi}{2}$. point 29 is at position $\left(\frac{-\pi}{4}, \frac{-3d\pi}{8}\right)$, point 30 is at position $\left(-\frac{\pi}{4}, \frac{-d\pi}{4}\right)$, point 36 is at position $\left(\frac{-3\pi}{16}, \frac{-d\pi}{8}\right)$, point 41 is at position $\left(\frac{-\pi}{8}, 0\right)$, point 45 is at position $\left(\frac{-\pi}{16}, 0\right)$, point 52 is at position $\left(\frac{\pi}{16}, 0\right)$, point 56 is at position $\left(\frac{\pi}{8}, 0\right)$, point 60 is at position $\left(\frac{3\pi}{16}, \frac{-d\pi}{8}\right)$, point 66 is at position $\left(\frac{\pi}{4}, \frac{-d\pi}{4}\right)$, point 65 is at position $\left(\frac{\pi}{4}, \frac{-3d\pi}{8}\right)$.

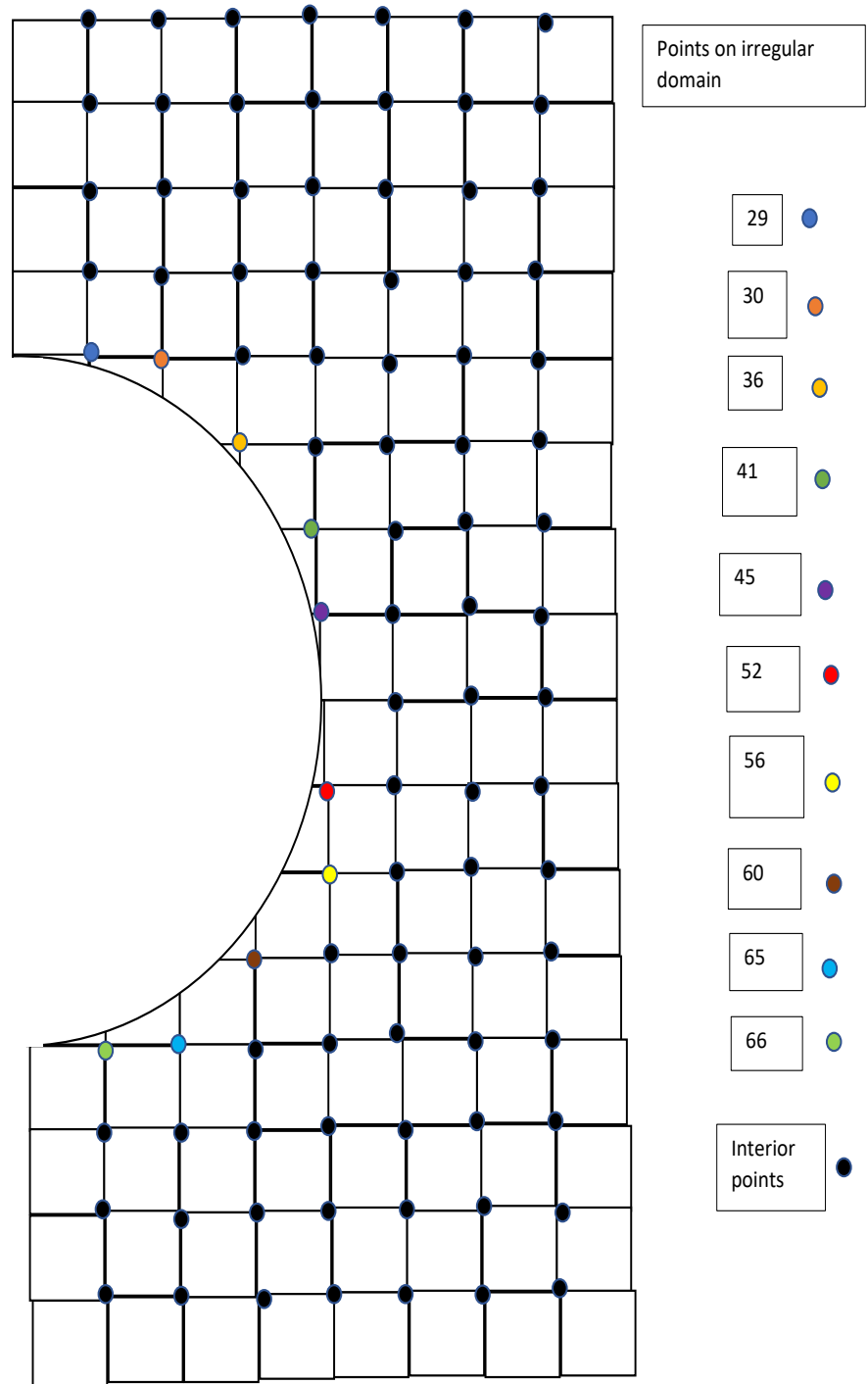


Figure 5.2: Irregular domain in 2D



5.2.1 The velocity u at irregular points is calculated below.

Velocity u at point 29

We form linear combination with coefficient C_1, C_2, C_3 to obtain the following

$$u_{xx} = C_1u(x - h, y) + C_2u(x, y) + C_3u(x + a_1h, y).$$

we use Taylor expansion series to obtain the following

$$u(x - h, y) = u - hu_x + \frac{1}{2}h^2u_{xx} + \dots$$

$$u(x + a_1h, y) = u + a_1hu_x + \frac{1}{2}(a_1h)^2u_{xx} + \dots$$

$$C_1(u - hu_x + \frac{1}{2}h^2u_{xx} + \dots) + C_2u + C_3(u + a_1hu_x + \frac{1}{2}(a_1h)^2u_{xx} + \dots) = u_{xx}$$

Then the approximation of 2^{nd} order partial derivative u_{xx} is given below

$$(C_1 + C_2 + C_3)u + (-hC_1 + a_1hC_3)u_x + (\frac{1}{2}h^2C_1 + \frac{1}{2}(a_1h)^2C_3)u_{xx} = u_{xx}.$$

$$C_1 + C_2 + C_3 = 0, \quad (5.19)$$

$$-hC_1 + a_1hC_3 = 0, \quad (5.20)$$

$$\frac{1}{2}h^2C_1 + \frac{1}{2}(a_1h)^2C_3 = 1. \quad (5.21)$$

From 5.20 $C_1 = a_1C_3$, substituting C_1 into 5.21

$$\frac{1}{2}h^2a_1C_3 + \frac{1}{2}(a_1h)^2C_3 = 1,$$

$$C_3 = \frac{2}{a_1h^2(a_1 + 1)},$$

$$C_1 = \frac{2}{h^2(a_1 + 1)}.$$

substituting C_3 and C_1 into 5.19

$$C_2 = -\frac{2}{a_1h^2(a_1 + 1)} - \frac{2}{h^2(a_1 + 1)} = -\frac{2}{ah^2}.$$



The results in the difference approximation for u_{xx} is given below

$$u_{xx} = \frac{2}{h^2(a_1 + 1)}u_{i-1,j} - \frac{2}{ah^2(a_1 + 1)}u_{i,j} - \frac{2}{ah^2}u_{i+1,j}.$$

Linear combination of u_{yy} with coefficient C_4, C_5, C_6 is given below

$$\begin{aligned} u_{yy} &= C_4u(x, y + h_1) + C_5u(x, y) + C_6u(x, y - h_1), \\ u(x, y + h_1) &= u + h_1u_y + \frac{1}{2}h_1^2u_{yy} + \dots \\ u(x, y - h_1) &= u - h_1u_y + \frac{1}{2}(h_1)^2u_{yy} + \dots \end{aligned}$$

We form linear combination with coefficient C_1, C_2, C_3 to obtain the following

$$\begin{aligned} C_4(u + h_1u_y + \frac{1}{2}h_1^2u_{yy} + \dots) + C_5u + C_6(u - h_1u_y + \frac{1}{2}(h_1)^2u_{yy} + \dots) &= u_{yy}, \\ (C_4 + C_5 + C_6)u + (h_1C_4 - h_1C_6)u_y + (\frac{1}{2}h_1^2C_4 + \frac{1}{2}(h_1)^2C_6)u_{yy} &= u_{yy}. \end{aligned}$$

$$C_4 + C_5 + C_6 = 0, \tag{5.22}$$

$$h_1C_4 - h_1C_6 = 0, \tag{5.23}$$

$$\frac{1}{2}h_1^2C_4 + \frac{1}{2}(h_1)^2C_6 = 1. \tag{5.24}$$

From 5.23 $C_4 = C_6$, substituting C_4 into 5.24

$$\frac{1}{2}h_1^2C_4 + \frac{1}{2}(h_1)^2C_4 = 1,$$

$$C_4 = \frac{1}{h_1^2},$$

$$C_6 = \frac{1}{h_1^2},$$

substituting C_4 and C_6 into 5.22

$$C_5 = -\frac{1}{h_1^2} - \frac{1}{h_1^2} = -\frac{2}{h_1^2}.$$

Results of difference approximation for u_{yy} is given below

$$u_{yy} = \frac{1}{h_1^2}u_{i,j+1} - \frac{2}{h_1^2}u_{i,j} + \frac{1}{h_1^2}u_{i,j-1}.$$

Velocity u at point 29

$$u_{i-1,j} - (a_1 + 1)\left(\frac{1}{a_1} + \lambda\right)u_{i,j} + \frac{1}{a_1}u_{i+1,j} - \frac{1}{2}\lambda(a_1 + 1)u_{i,j+1} + \frac{1}{2}\lambda(a_1 + 1)u_{i,j-1} = 0. \tag{5.25}$$

Velocity u at point 65

$$u_{i+1,j} - (a_1 + 1)\left(\frac{1}{a_1} + \lambda\right)u_{i,j} + \frac{1}{a_1}u_{i-1,j} - \frac{1}{2}\lambda(a_1 + 1)u_{i,j+1} + \frac{1}{2}\lambda(a_1 + 1)u_{i,j-1} = 0. \tag{5.26}$$



Velocity u at point 30 Linear combination of u_{xx} with coefficient C_1, C_2, C_3 is given below

$$\begin{aligned} u_{xx} &= C_1 u(x-h, y) + C_2 u(x, y) + C_3 u(x+a_2 h, y), \\ u(x-h, y) &= u - hu_x + \frac{1}{2} h^2 u_{xx} + \dots, \\ u(x+a_2 h, y) &= u + a_2 h u_x + \frac{1}{2} (a_2 h)^2 u_{xx} + \dots \end{aligned}$$

We form linear combination with coefficient C_1, C_2, C_3 to obtain the following

$$C_1(u - hu_x + \frac{1}{2} h^2 u_{xx} + \dots) + C_2 u + C_3(u + a_2 h u_x + \frac{1}{2} (a_2 h)^2 u_{xx} + \dots) = u_{xx},$$

$$(C_1 + C_2 + C_3)u + (-hC_1 + a_2 h C_3)u_x + (\frac{1}{2} h^2 C_1 + \frac{1}{2} (a_2 h)^2 C_3)u_{xx} = u_{xx},$$

$$C_1 + C_2 + C_3 = 0, \quad (5.27)$$

$$-hC_1 + a_2 h C_3 = 0, \quad (5.28)$$

$$\frac{1}{2} h^2 C_1 + \frac{1}{2} (a_2 h)^2 C_3 = 1, \quad (5.29)$$

From 5.20 $C_1 = a_2 C_3$, substituting C_1 into 5.29

$$\frac{1}{2} h^2 a_2 C_3 + \frac{1}{2} (a_2 h)^2 C_3 = 1,$$

$$C_3 = \frac{2}{a_1 h^2 (a_2 + 1)},$$

$$C_1 = \frac{2}{h^2 (a_2 + 1)},$$

substituting C_3 and C_1 into 5.27

$$C_2 = -\frac{2}{a_2 h^2 (a_1 + 1)} - \frac{2}{h^2 (a_2 + 1)} = -\frac{2}{a_2 h^2}.$$

Results of difference approximation for u_{xx} is given below

$$u_{xx} = \frac{2}{h^2 (a_2 + 1)} u_{i-1,j} - \frac{2}{a_2 h^2 (a_2 + 1)} u_{i,j} + \frac{2}{a_2 h^2 (a_2 + 1)} u_{i+1,j},$$

Linear combination of u_{yy} with coefficient C_4, C_5, C_6 is given below

$$\begin{aligned} u_{yy} &= C_4 u(x, y+h_1) + C_5 u(x, y) + C_6 u(x, y-h_1), \\ u(x, y+h_1) &= u + h_1 u_y + \frac{1}{2} h_1^2 u_{yy} + \dots \\ u(x, y-h_1) &= u - h_1 u_y + \frac{1}{2} (h_1)^2 u_{yy} + \dots \end{aligned}$$



We form linear combination with coefficient C_1, C_2, C_3 to obtain the following

$$C_4(u + h_1u_y + \frac{1}{2}h_1^2u_{yy} + \dots) + C_5u + C_6(u - h_1u_y + \frac{1}{2}(h_1)^2u_{yy} + \dots) = u_{yy},$$

$$(C_4 + C_5 + C_6)u + (h_1C_4 - h_1C_6)u_y + (\frac{1}{2}h_1^2C_4 + \frac{1}{2}(h_1)^2C_6)u_{yy} = u_{yy},$$

$$C_4 + C_5 + C_6 = 0, \quad (5.30)$$

$$h_1C_4 - h_1C_6 = 0, \quad (5.31)$$

$$\frac{1}{2}h_1^2C_4 + \frac{1}{2}(h_1)^2C_6 = 1, \quad (5.32)$$

From 5.31 $C_4 = C_6$, substituting C_4 into 5.32

$$\frac{1}{2}h_1^2C_4 + \frac{1}{2}(h_1)^2C_4 = 1,$$

$$C_4 = \frac{1}{h_1^2},$$

$$C_6 = \frac{1}{h_1^2},$$

substituting C_4 and C_6 into 5.30

$$C_5 = -\frac{1}{h_1^2} - \frac{1}{h_1^2} = -\frac{2}{h_1^2},$$

Results of difference approximation for u_{xx} is given below

$$u_{yy} = \frac{1}{h_1^2}u_{i,j+1} - \frac{2}{h_1^2}u_{i,j} + \frac{1}{h_1^2}u_{i,j-1}.$$

Velocity u at point 30

$$u_{i-1,j} - (a_2 + 1)\left(\frac{1}{a_2} + \lambda\right)u_{i,j} + \frac{1}{a_2}u_{i+1,j} - \frac{1}{2}\lambda(a_2 + 1)u_{i,j+1} + \frac{1}{2}\lambda(a_2 + 1)u_{i,j-1} = 0. \quad (5.33)$$

Velocity u at point 66

$$u_{i+1,j} - (a_2 + 1)\left(\frac{1}{a_2} + \lambda\right)u_{i,j} + \frac{1}{a_2}u_{i-1,j} - \frac{1}{2}\lambda(a_2 + 1)u_{i,j+1} + \frac{1}{2}\lambda(a_2 + 1)u_{i,j-1} = 0. \quad (5.34)$$

Velocity u at point 36 Linear combination of u_{xx} with coefficient C_1, C_2, C_3 is given below

$$u_{xx} = C_1u(x - h, y) + C_2u(x, y) + C_3u(x + a_3h, y),$$

$$u(x - h, y) = u - hu_x + \frac{1}{2}h^2u_{xx} + \dots$$

$$u(x + a_3h, y) = u + a_3hu_x + \frac{1}{2}(a_3h)^2u_{xx} + \dots$$



We form linear combination with coefficient C_1, C_2, C_3 to obtain the following

$$C_1(u - hu_x + \frac{1}{2}h^2u_{xx} + \dots) + C_2u + C_3(u + a_3hu_x + \frac{1}{2}(a_3h)^2u_{xx} + \dots) = u_{xx}$$

$$(C_1 + C_2 + C_3)u + (-hC_1 + a_3hC_3)u_x + (\frac{1}{2}h^2C_1 + \frac{1}{2}(a_3h)^2C_3)u_{xx} = u_{xx}$$

$$C_1 + C_2 + C_3 = 0, \quad (5.35)$$

$$-hC_1 + a_3hC_3 = 0, \quad (5.36)$$

$$\frac{1}{2}h^2C_1 + \frac{1}{2}(a_3h)^2C_3 = 1, \quad (5.37)$$

From 5.36 $C_1 = a_3C_3$, substituting C_1 into 5.37

$$\frac{1}{2}h^2a_3C_3 + \frac{1}{2}(a_3h)^2C_3 = 1,$$

$$C_3 = \frac{2}{a_3h^2(a_3 + 1)},$$

$$C_1 = \frac{2}{h^2(a_3 + 1)},$$

substituting C_3 and C_1 into 5.35

$$C_2 = -\frac{2}{a_3h^2(a_3 + 1)} - \frac{2}{h^2(a_3 + 1)} = -\frac{2}{a_3h^2},$$

Results of difference approximation for u_{xx} is given below

$$u_{xx} = \frac{2}{h^2(a_3 + 1)}u_{i-1,j} - \frac{2}{a_3h^2(a_3 + 1)}u_{i,j} + \frac{2}{a_3h^2(a_3 + 1)}u_{i+1,j}.$$

Linear combination of u_{yy} with coefficient C_4, C_5, C_6 is given below

$$u_{yy} = C_4u(x, y + h_1) + C_5u(x, y) + C_6u(x, y - a_4h_1),$$

$$u(x, y + h_1) = u + h_1u_y + \frac{1}{2}h_1^2u_{yy} + \dots$$

$$u(x, y - a_4h_1) = u - a_4h_1u_y + \frac{1}{2}(a_4h_1)^2u_{yy} + \dots$$

We form linear combination with coefficient C_1, C_2, C_3 to obtain the following

$$C_4(u + h_1u_y + \frac{1}{2}h_1^2u_{yy} + \dots) + C_5u + C_6(u - a_4h_1u_y + \frac{1}{2}(a_4h_1)^2u_{yy} + \dots) = u_{yy},$$

$$(C_4 + C_5 + C_6)u + (h_1C_4 - a_4h_1C_6)u_y + \left(\frac{1}{2}h_1^2C_4 + \frac{1}{2}(a_4h_1)^2C_6\right)u_{yy} = u_{yy},$$

$$C_4 + C_5 + C_6 = 0, \quad (5.38)$$

$$h_1C_4 - a_4h_1C_6 = 0, \quad (5.39)$$

$$\frac{1}{2}h_1^2C_4 + \frac{1}{2}(a_4h_1)^2C_6 = 1, \quad (5.40)$$

From 5.23 $C_4 = a_4C_6$, substituting C_4 into 5.24

$$\frac{1}{2}a_4h_1^2C_6 + \frac{1}{2}(a_4h_1)^2C_6 = 1,$$

$$C_4 = \frac{2}{h_1^2(1 + a_4)},$$

$$C_6 = \frac{2}{a_4h_1^2(1 + a_4)},$$

substituting C_4 and C_6 into 5.22

$$C_5 = -\frac{1}{h_1^2(1 + a_4)} - \frac{1}{a_4h_1^2(1 + a_4)} = -\frac{2}{a_4h_1^2},$$

Results of difference approximation for u_{yy} is given below

$$u_{yy} = \frac{2}{h_1^2(1 + a_4)}u_{i,j+1} - \frac{2}{a_4h_1^2}u_{i,j} + \frac{1}{a_4h_1^2}u_{i,j-1}.$$

Velocity u at point 36

$$u_{i-1,j} - (a_2 + 1)\left(\frac{1}{a_2} + \lambda\right)u_{i,j} + \frac{1}{a_2}u_{i+1,j} - \frac{1}{2}\lambda(a_2 + 1)u_{i,j+1} + \frac{1}{2}\lambda(a_2 + 1)u_{i,j-1} = 0. \quad (5.41)$$

Velocity u at point 60

$$u_{i+1,j} - (a_2 + 1)\left(\frac{1}{a_2} + \lambda\right)u_{i,j} + \frac{1}{a_2}u_{i-1,j} - \frac{1}{2}\lambda(a_2 + 1)u_{i,j+1} + \frac{1}{2}\lambda(a_2 + 1)u_{i,j-1} = 0. \quad (5.42)$$

Velocity u at point 41 Linear combination of u_{xx} with coefficient C_1, C_2, C_3 is given below

$$\begin{aligned} u_{xx} &= C_1u(x - h, y) + C_2u(x, y) + C_3u(x + h, y), \\ u(x - h, y) &= u - hu_x + \frac{1}{2}h^2u_{xx} + \dots \\ u(x + h, y) &= u + hu_x + \frac{1}{2}(h)^2u_{xx} + \dots \end{aligned}$$



We form linear combination with coefficient C_1, C_2, C_3 to obtain the following

$$C_1(u - hu_x + \frac{1}{2}h^2u_{xx} + \dots) + C_2u + C_3(u + hu_x + \frac{1}{2}(h)^2u_{xx} + \dots) = u_{xx},$$

$$(C_1 + C_2 + C_3)u + (-hC_1 + hC_3)u_x + (\frac{1}{2}h^2C_1 + \frac{1}{2}(h)^2C_3)u_{xx} = u_{xx},$$

$$C_1 + C_2 + C_3 = 0, \quad (5.43)$$

$$-hC_1 + hC_3 = 0, \quad (5.44)$$

$$\frac{1}{2}h^2C_1 + \frac{1}{2}(h)^2C_3 = 1, \quad (5.45)$$

From 5.44 $C_1 = C_3$, substituting C_1 into 5.45

$$\frac{1}{2}h^2C_3 + \frac{1}{2}(h)^2C_3 = 1,$$

$$C_3 = \frac{1}{h^2},$$

$$C_1 = \frac{2}{h^2},$$

substituting C_3 and C_1 into 5.43

$$C_2 = -\frac{1}{h^2} - \frac{2}{h^2} = -\frac{3}{h^2},$$

Results of difference approximation for u_{xx} is given below

$$u_{xx} = \frac{1}{h^2}u_{i-1,j} - \frac{2}{h^2}u_{i,j} + \frac{1}{h^2}u_{i+1,j}.$$

Linear combination of u_{yy} with coefficient C_4, C_5, C_6 is given below

$$u_{yy} = C_4u(x, y + h_1) + C_5u(x, y) + C_6u(x, y - a_4h_1),$$

$$u(x, y + h_1) = u + h_1u_y + \frac{1}{2}h_1^2u_{yy} + \dots$$

$$u(x, y - a_5h_1) = u - a_5h_1u_y + \frac{1}{2}(a_5h_1)^2u_{yy} + \dots$$

We form linear combination with coefficient C_1, C_2, C_3 to obtain the following

$$C_4(u + h_1u_y + \frac{1}{2}h_1^2u_{yy} + \dots) + C_5u + C_6(u - a_5h_1u_y + \frac{1}{2}(a_5h_1)^2u_{yy} + \dots) = u_{yy},$$

$$(C_4 + C_5 + C_6)u + (h_1C_5 - a_4h_1C_6)u_y + \left(\frac{1}{2}h_1^2C_4 + \frac{1}{2}(a_5h_1)^2C_6\right)u_{yy} = u_{yy},$$

$$C_4 + C_5 + C_6 = 0, \quad (5.46)$$

$$h_1C_4 - a_5h_1C_6 = 0, \quad (5.47)$$

$$\frac{1}{2}h_1^2C_4 + \frac{1}{2}(a_5h_1)^2C_6 = 1, \quad (5.48)$$

From 5.47 $C_4 = a_5C_6$, substituting C_4 into 5.48

$$\frac{1}{2}a_5h_1^2C_6 + \frac{1}{2}(a_5h_1)^2C_6 = 1,$$

$$C_4 = \frac{2}{h_1^2(1 + a_5)},$$

$$C_6 = \frac{2}{a_5h_1^2(1 + a_4)},$$

substituting C_4 and C_6 into 5.46

$$C_5 = -\frac{1}{h_1^2(1 + a_5)} - \frac{1}{a_5h_1^2(1 + a_5)} = -\frac{2}{a_5h_1^2},$$

Results of difference approximation for u_{yy} is given below

$$u_{yy} = \frac{2}{h_1^2(1 + a_5)}u_{i,j+1} - \frac{2}{a_5h_1^2}u_{i,j} + \frac{1}{a_5h_1^2(1 + a_5)}u_{i,j-1}.$$

Velocity u at point 41

$$u_{i-1,j} - (a_5 + 1)\left(\frac{1}{a_5} + \lambda\right)u_{i,j} + \frac{1}{a_5}u_{i+1,j} - \frac{1}{2}\lambda(a_5 + 1)u_{i,j+1} + \frac{1}{2}\lambda(a_5 + 1)u_{i,j-1} = 0. \quad (5.49)$$

Velocity u at point 56

$$u_{i+1,j} - (a_5 + 1)\left(\frac{1}{a_5} + \lambda\right)u_{i,j} + \frac{1}{a_5}u_{i-1,j} - \frac{1}{2}\lambda(a_5 + 1)u_{i,j+1} + \frac{1}{2}\lambda(a_5 + 1)u_{i,j-1} = 0. \quad (5.50)$$

Velocity u at point 45 Linear combination of u_{xx} with coefficient C_1, C_2, C_3 is given below

$$u_{xx} = C_1u(x - h, y) + C_2u(x, y) + C_3u(x + h, y),$$

$$u(x - h, y) = u - hu_x + \frac{1}{2}h^2u_{xx} + \dots$$

$$u(x + h, y) = u + hu_x + \frac{1}{2}(h)^2u_{xx} + \dots$$



We form linear combination with coefficient C_1, C_2, C_3 to obtain the following

$$C_1(u - hu_x + \frac{1}{2}h^2u_{xx} + \dots) + C_2u + C_3(u + hu_x + \frac{1}{2}(h)^2u_{xx} + \dots) = u_{xx},$$

$$(C_1 + C_2 + C_3)u + (-hC_1 + hC_3)u_x + (\frac{1}{2}h^2C_1 + \frac{1}{2}(h)^2C_3)u_{xx} = u_{xx},$$

$$C_1 + C_2 + C_3 = 0, \quad (5.51)$$

$$-hC_1 + hC_3 = 0, \quad (5.52)$$

$$\frac{1}{2}h^2C_1 + \frac{1}{2}(h)^2C_3 = 1, \quad (5.53)$$

From 5.52 $C_1 = C_3$, substituting C_1 into 5.53

$$\frac{1}{2}h^2C_3 + \frac{1}{2}(h)^2C_3 = 1,$$

$$C_3 = \frac{1}{h^2},$$

$$C_1 = \frac{2}{h^2},$$

substituting C_3 and C_1 into 5.51

$$C_2 = -\frac{1}{h^2} - \frac{2}{h^2} = -\frac{3}{h^2},$$

Results of difference approximation for u_{xx} is given below

$$u_{xx} = \frac{1}{h^2}U_{i-1,j} - \frac{2}{h^2}u_{i,j} + \frac{1}{h^2}u_{i+1,j}.$$

Linear combination of u_{yy} with coefficient C_4, C_5, C_6 is given below

$$u_{yy} = C_4u(x, y + h_1) + C_5u(x, y) + C_6u(x, y - a_6h_1),$$

$$u(x, y + h_1) = u + h_1u_y + \frac{1}{2}h_1^2u_{yy} + \dots$$

$$u(x, y - a_6h_1) = u - a_6h_1u_y + \frac{1}{2}(a_6h_1)^2u_{yy} + \dots$$

$$C_4(u + h_1u_y + \frac{1}{2}h_1^2u_{yy} + \dots) + C_5u + C_6(u - a_6h_1u_y + \frac{1}{2}(a_6h_1)^2u_{yy} + \dots) = u_{yy}$$



We form linear combination with coefficient C_1, C_2, C_3 to obtain the following

$$(C_4 + C_5 + C_6)u + (h_1C_5 - a_6h_1C_6)u_y + \left(\frac{1}{2}h_1^2C_4 + \frac{1}{2}(a_6h_1)^2C_6\right)u_{yy} = u_{yy}$$

$$C_4 + C_5 + C_6 = 0, \tag{5.54}$$

$$h_1C_4 - a_6h_1C_6 = 0, \tag{5.55}$$

$$\frac{1}{2}h_1^2C_4 + \frac{1}{2}(a_6h_1)^2C_6 = 1, \tag{5.56}$$

From 5.55 $C_4 = a_6C_6$, substituting C_4 into 5.56

$$\frac{1}{2}a_6h_1^2C_6 + \frac{1}{2}(a_6h_1)^2C_6 = 1,$$

$$C_4 = \frac{2}{h_1^2(1 + a_6)},$$

$$C_6 = \frac{2}{a_6h_1^2(1 + a_6)},$$

substituting C_4 and C_6 into 5.54

$$C_5 = -\frac{1}{h_1^2(1 + a_6)} - \frac{1}{a_6h_1^2(1 + a_6)} = -\frac{2}{a_6h_1^2},$$

Results of difference approximation for u_{yy} is given below

$$u_{yy} = \frac{2}{h_1^2(1 + a_6)}u_{i,j+1} - \frac{2}{a_6h_1^2}u_{i,j} + \frac{1}{a_6h_1^2(1 + a_6)}u_{i,j-1}.$$

Velocity u at point 45

$$u_{i-1,j} - (a_6 + 1)\left(\frac{1}{a_6} + \lambda\right)u_{i,j} + \frac{1}{a_6}u_{i+1,j} - \frac{1}{2}\lambda(a_6 + 1)u_{i,j+1} + \frac{1}{2}\lambda(a_6 + 1)u_{i,j-1} = 0. \tag{5.57}$$

Velocity u at point 52

$$u_{i+1,j} - (a_6 + 1)\left(\frac{1}{a_6} + \lambda\right)u_{i,j} + \frac{1}{a_6}u_{i-1,j} - \frac{1}{2}\lambda(a_6 + 1)u_{i,j+1} + \frac{1}{2}\lambda(a_6 + 1)u_{i,j-1} = 0. \tag{5.58}$$

The discretisation of the velocity u using boundary conditions given below

$$u|_{\Omega_{ver}} = 0, u|_{\Omega_{hor}} = 1.$$

thus

$$\begin{aligned} u &= 0 & \text{for } & x = 0 & \text{and } & x = N_x \\ u &= 1 & \text{for } & y = 0 & \text{and } & y = N_y \end{aligned}$$

$$u_{0,j} = 0, u_{N_x,j} = 0, u_{i,0} = 1 \text{ and } u_{i,N_y} = 1$$



5.2.2 The velocity v at irregular points is calculated below.

Linear combination of v_{xx} with coefficient C_1, C_2, C_3 is given below

$$v_{xx} = C_1v(x - h, y) + C_2v(x, y) + C_3v(x + a_1h, y),$$

$$v(x - h, y) = v - hv_x + \frac{1}{2}h^2v_{xx} + \dots$$

$$v(x + a_1h, y) = v + a_1hv_x + \frac{1}{2}(a_1h)^2v_{xx} + \dots$$

We form linear combination with coefficient C_1, C_2, C_3 to obtain the following

$$C_1(v - hv_x + \frac{1}{2}h^2v_{xx} + \dots) + C_2v + C_3(v + a_1hv_x + \frac{1}{2}(a_1h)^2v_{xx} + \dots) = v_{xx},$$

$$(C_1 + C_2 + C_3)v + (-hC_1 + a_1hC_3)v_x + (\frac{1}{2}h^2C_1 + \frac{1}{2}(a_1h)^2C_3)v_{xx} = v_{xx},$$

$$C_1 + C_2 + C_3 = 0, \quad (5.59)$$

$$-hC_1 + a_1hC_3 = 0, \quad (5.60)$$

$$\frac{1}{2}h^2C_1 + \frac{1}{2}(a_1h)^2C_3 = 1, \quad (5.61)$$

From 5.60 $C_1 = a_1C_3$, substituting C_1 into 5.61

$$\frac{1}{2}h^2a_1C_3 + \frac{1}{2}(a_1h)^2C_3 = 1,$$

$$C_3 = \frac{2}{a_1h^2(a_1 + 1)},$$

$$C_1 = \frac{2}{h^2(a_1 + 1)},$$

substituting C_3 and C_1 into 5.59

$$C_2 = -\frac{2}{a_1h^2(a_1 + 1)} - \frac{2}{h^2(a_1 + 1)} = -\frac{2}{ah^2},$$

Results of difference approximation for u_{xx} is given below

$$v_{xx} = \frac{2}{h^2(a_1 + 1)} - \frac{2}{ah^2(a_1 + 1)} = -\frac{2}{ah^2}.$$



Linear combination of v_{yy} with coefficient C_4, C_5, C_6 is given below

$$\begin{aligned}v_{yy} &= C_4v(x, y + h_1) + C_5v(x, y) + C_6v(x, y - h_1), \\v(x, y + h_1) &= v + h_1v_y + \frac{1}{2}h_1^2v_{yy} + \dots \\v(x, y - h_1) &= v - h_1v_y + \frac{1}{2}(h_1)^2v_{yy} + \dots\end{aligned}$$

We form linear combination with coefficient C_1, C_2, C_3 to obtain the following

$$C_4(u + h_1v_y + \frac{1}{2}h_1^2v_{yy} + \dots) + C_5v + C_6(v - h_1v_y + \frac{1}{2}(h_1)^2v_{yy} + \dots) = v_{yy},$$

$$(C_4 + C_5 + C_6)v + (h_1C_4 - h_1C_6)v_y + (\frac{1}{2}h_1^2C_4 + \frac{1}{2}(h_1)^2C_6)v_{yy} = v_{yy},$$

$$C_4 + C_5 + C_6 = 0, \quad (5.62)$$

$$h_1C_4 - h_1C_6 = 0, \quad (5.63)$$

$$\frac{1}{2}h_1^2C_4 + \frac{1}{2}(h_1)^2C_6 = 1, \quad (5.64)$$

From 5.63 $C_4 = C_6$, substituting C_4 into 5.64

$$\frac{1}{2}h_1^2C_4 + \frac{1}{2}(h_1)^2C_4 = 1,$$

$$C_4 = \frac{1}{h_1^2},$$

$$C_6 = \frac{1}{h_1^2},$$

substituting C_4 and C_6 into 5.62

$$C_5 = -\frac{1}{h_1^2} - \frac{1}{h_1^2} = -\frac{2}{h_1^2},$$

Results of difference approximation for u_{yy} is given below

$$v_{yy} = \frac{1}{h_1^2}v_{i,j+1} - \frac{2}{h_1^2}v_{i,j} + \frac{1}{h_1^2}v_{i,j-1}.$$

Velocity v at point 29

$$v_{i-1,j} - (a_1 + 1)\left(\frac{1}{a_1} + \lambda\right)v_{i,j} + \frac{1}{a_1}v_{i+1,j} - \frac{1}{2}\lambda(a_1 + 1)v_{i,j+1} + \frac{1}{2}\lambda(a_1 + 1)v_{i,j-1} = 0. \quad (5.65)$$

Velocity v at point 65

$$v_{i+1,j} - (a_1 + 1)\left(\frac{1}{a_1} + \lambda\right)v_{i,j} + \frac{1}{a_1}v_{i-1,j} - \frac{1}{2}\lambda(a_1 + 1)v_{i,j+1} + \frac{1}{2}\lambda(a_1 + 1)v_{i,j-1} = 0. \quad (5.66)$$

Velocity v at point 30 Linear combination of v_{xx} with coefficient C_1, C_2, C_3 is given below

$$\begin{aligned} v_{xx} &= C_1v(x - h, y) + C_2v(x, y) + C_3v(x + a_2h, y), \\ v(x - h, y) &= v - hv_x + \frac{1}{2}h^2v_{xx} + \dots \\ v(x + a_2h, y) &= v + a_2hv_x + \frac{1}{2}(a_2h)^2v_{xx} + \dots \end{aligned}$$

We form linear combination with coefficient C_1, C_2, C_3 to obtain the following

$$C_1(v - hv_x + \frac{1}{2}h^2v_{xx} + \dots) + C_2v + C_3(v + a_2hv_x + \frac{1}{2}(a_2h)^2v_{xx} + \dots) = v_{xx},$$

$$(C_1 + C_2 + C_3)v + (-hC_1 + a_2hC_3)v_x + (\frac{1}{2}h^2C_1 + \frac{1}{2}(a_2h)^2C_3)v_{xx} = v_{xx},$$

$$C_1 + C_2 + C_3 = 0, \quad (5.67)$$

$$-hC_1 + a_2hC_3 = 0, \quad (5.68)$$

$$\frac{1}{2}h^2C_1 + \frac{1}{2}(a_2h)^2C_3 = 1, \quad (5.69)$$

From 5.60 $C_1 = a_2C_3$, substituting C_1 into 5.69

$$\frac{1}{2}h^2a_2C_3 + \frac{1}{2}(a_2h)^2C_3 = 1,$$

$$C_3 = \frac{2}{a_1h^2(a_2 + 1)},$$

$$C_1 = \frac{2}{h^2(a_2 + 1)},$$

substituting C_3 and C_1 into 5.67

$$C_2 = -\frac{2}{a_2h^2(a_1 + 1)} - \frac{2}{h^2(a_2 + 1)} = -\frac{2}{a_2h^2},$$

Results of difference approximation for u_{xx} is given below

$$v_{xx} = \frac{2}{h^2(a_2 + 1)}v_{i-1,j} - \frac{2}{a_2h^2(a_2 + 1)}v_{i,j} + \frac{2}{a_2h^2(a_2 + 1)}v_{i+1,j}.$$



Linear combination of v_{yy} with coefficient C_4, C_5, C_6 is given below

$$\begin{aligned}v_{yy} &= C_4 v(x, y + h_1) + C_5 v(x, y) + C_6 v(x, y - h_1), \\v(x, y + h_1) &= v + h_1 u_y + \frac{1}{2} h_1^2 v_{yy} + \dots \\v(x, y - h_1) &= v - h_1 u_y + \frac{1}{2} (h_1)^2 v_{yy} + \dots\end{aligned}$$

We form linear combination with coefficient C_1, C_2, C_3 to obtain the following

$$C_4(v + h_1 v_y + \frac{1}{2} h_1^2 v_{yy} + \dots) + C_5 v + C_6(v - h_1 v_y + \frac{1}{2} (h_1)^2 v_{yy} + \dots) = v_{yy},$$

$$(C_4 + C_5 + C_6)v + (h_1 C_4 - h_1 C_6)v_y + (\frac{1}{2} h_1^2 C_4 + \frac{1}{2} (h_1)^2 C_6)v_{yy} = v_{yy},$$

$$C_4 + C_5 + C_6 = 0, \quad (5.70)$$

$$h_1 C_4 - h_1 C_6 = 0, \quad (5.71)$$

$$\frac{1}{2} h_1^2 C_4 + \frac{1}{2} (h_1)^2 C_6 = 1, \quad (5.72)$$

From 5.71 $C_4 = C_6$, substituting C_4 into 5.72

$$\frac{1}{2} h_1^2 C_4 + \frac{1}{2} (h_1)^2 C_4 = 1,$$

$$C_4 = \frac{1}{h_1^2},$$

$$C_6 = \frac{1}{h_1^2},$$

substituting C_4 and C_6 into 5.70

$$C_5 = -\frac{1}{h_1^2} - \frac{1}{h_1^2} = -\frac{2}{h_1^2},$$

Results of difference approximation for u_{xx} is given below

$$v_{yy} = \frac{1}{h_1^2} v_{i,j+1} - \frac{2}{h_1^2} v_{i,j} + \frac{1}{h_1^2} v_{i,j-1}.$$

Velocity v at point 30

$$v_{i-1,j} - (a_2 + 1)\left(\frac{1}{a_2} + \lambda\right)v_{i,j} + \frac{1}{a_2} v_{i+1,j} - \frac{1}{2} \lambda (a_2 + 1) v_{i,j+1} + \frac{1}{2} \lambda (a_2 + 1) v_{i,j-1} = 0. \quad (5.73)$$



Velocity v at point 66

$$v_{i+1,j} - (a_2 + 1)\left(\frac{1}{a_2} + \lambda\right)v_{i,j} + \frac{1}{a_2}v_{i-1,j} - \frac{1}{2}\lambda(a_2 + 1)v_{i,j+1} + \frac{1}{2}\lambda(a_2 + 1)v_{i,j-1} = 0. \quad (5.74)$$

Velocity v at point 36 Linear combination of v_{xx} with coefficient C_1, C_2, C_3 is given below

$$\begin{aligned} v_{xx} &= C_1v(x - h, y) + C_2v(x, y) + C_3v(x + a_3h, y), \\ v(x - h, y) &= v - hv_x + \frac{1}{2}h^2v_{xx} + \dots\dots \\ v(x + a_3h, y) &= v + a_3hv_x + \frac{1}{2}(a_3h)^2v_{xx} + \dots \end{aligned}$$

We form linear combination with coefficient C_1, C_2, C_3 to obtain the following

$$C_1(v - hv_x + \frac{1}{2}h^2v_{xx} + \dots) + C_2v + C_3(v + a_3hv_x + \frac{1}{2}(a_3h)^2v_{xx} + \dots) = v_{xx},$$

$$(C_1 + C_2 + C_3)v + (-hC_1 + a_3hC_3)v_x + (\frac{1}{2}h^2C_1 + \frac{1}{2}(a_3h)^2C_3)v_{xx} = v_{xx},$$

$$C_1 + C_2 + C_3 = 0, \quad (5.75)$$

$$-hC_1 + a_3hC_3 = 0, \quad (5.76)$$

$$\frac{1}{2}h^2C_1 + \frac{1}{2}(a_3h)^2C_3 = 1, \quad (5.77)$$

From 5.76 $C_1 = a_3C_3$, substituting C_1 into 5.77

$$\frac{1}{2}h^2a_3C_3 + \frac{1}{2}(a_3h)^2C_3 = 1,$$

$$C_3 = \frac{2}{a_3h^2(a_3 + 1)},$$

$$C_1 = \frac{2}{h^2(a_3 + 1)},$$

substituting C_3 and C_1 into 5.75

$$C_2 = -\frac{2}{a_3h^2(a_3 + 1)} - \frac{2}{h^2(a_3 + 1)} = -\frac{2}{a_3h^2},$$

Results of difference approximation for u_{xx} is given below

$$v_{xx} = \frac{2}{h^2(a_3 + 1)}v_{i-1,j} - \frac{2}{a_3h^2(a_3 + 1)}v_{i,j} + \frac{2}{a_3h^2(a_3 + 1)}v_{i+1,j}.$$



Linear combination of v_{yy} with coefficient C_4, C_5, C_6 is given below

$$\begin{aligned}v_{yy} &= C_4 v(x, y + h_1) + C_5 v(x, y) + C_6 v(x, y - a_4 h_1), \\v(x, y + h_1) &= v + h_1 v_y + \frac{1}{2} h_1^2 v_{yy} + \dots \\v(x, y - a_4 h_1) &= v - a_4 h_1 v_y + \frac{1}{2} (a_4 h_1)^2 v_{yy} + \dots\end{aligned}$$

We form linear combination with coefficient C_1, C_2, C_3 to obtain the following

$$C_4(v + h_1 v_y + \frac{1}{2} h_1^2 v_{yy} + \dots) + C_5 v + C_6(v - a_4 h_1 v_y + \frac{1}{2} (a_4 h_1)^2 v_{yy} + \dots) = v_{yy},$$

$$(C_4 + C_5 + C_6)v + (h_1 C_4 - a_4 h_1 C_6)v_y + (\frac{1}{2} h_1^2 C_4 + \frac{1}{2} (a_4 h_1)^2 C_6)v_{yy} = v_{yy},$$

$$C_4 + C_5 + C_6 = 0, \quad (5.78)$$

$$h_1 C_4 - a_4 h_1 C_6 = 0, \quad (5.79)$$

$$\frac{1}{2} h_1^2 C_4 + \frac{1}{2} (a_4 h_1)^2 C_6 = 1, \quad (5.80)$$

From 5.63 $C_4 = a_4 C_6$, substituting C_4 into 5.64

$$\frac{1}{2} a_4 h_1^2 C_6 + \frac{1}{2} (a_4 h_1)^2 C_6 = 1,$$

$$C_4 = \frac{2}{h_1^2(1 + a_4)},$$

$$C_6 = \frac{2}{a_4 h_1^2(1 + a_4)},$$

substituting C_4 and C_6 into 5.62

$$C_5 = -\frac{1}{h_1^2(1 + a_4)} - \frac{1}{a_4 h_1^2(1 + a_4)} = -\frac{2}{a_4 h_1^2},$$

Results of difference approximation for u_{yy} is given below

$$v_{yy} = \frac{2}{h_1^2(1 + a_4)} v_{i,j+1} - \frac{2}{a_4 h_1^2} v_{i,j} + \frac{1}{a_4 h_1^2} v_{i,j-1}.$$

Velocity v at point 36

$$v_{i-1,j} - (a_2 + 1)\left(\frac{1}{a_2} + \lambda\right)v_{i,j} + \frac{1}{a_2} v_{i+1,j} - \frac{1}{2} \lambda (a_2 + 1) v_{i,j+1} + \frac{1}{2} \lambda (a_2 + 1) v_{i,j-1} = 0. \quad (5.81)$$



Velocity v at point 60

$$v_{i+1,j} - (a_2 + 1)\left(\frac{1}{a_2} + \lambda\right)v_{i,j} + \frac{1}{a_2}v_{i-1,j} - \frac{1}{2}\lambda(a_2 + 1)v_{i,j+1} + \frac{1}{2}\lambda(a_2 + 1)v_{i,j-1} = 0. \quad (5.82)$$

Velocity v at point 41 Linear combination of v_{xx} with coefficient C_1, C_2, C_3 is given below

$$\begin{aligned} v_{xx} &= C_1v(x - h, y) + C_2v(x, y) + C_3v(x + h, y), \\ v(x - h, y) &= v - hv_x + \frac{1}{2}h^2v_{xx} + \dots \\ v(x + h, y) &= v + hv_x + \frac{1}{2}(h)^2v_{xx} + \dots \end{aligned}$$

We form linear combination with coefficient C_1, C_2, C_3 to obtain the following

$$C_1\left(v - hv_x + \frac{1}{2}h^2v_{xx} + \dots\right) + C_2v + C_3\left(v + hv_x + \frac{1}{2}(h)^2v_{xx} + \dots\right) = v_{xx},$$

$$(C_1 + C_2 + C_3)v + (-hC_1 + hC_3)v_x + \left(\frac{1}{2}h^2C_1 + \frac{1}{2}(h)^2C_3\right)v_{xx} = v_{xx},$$

$$C_1 + C_2 + C_3 = 0, \quad (5.83)$$

$$-hC_1 + hC_3 = 0, \quad (5.84)$$

$$\frac{1}{2}h^2C_1 + \frac{1}{2}(h)^2C_3 = 1, \quad (5.85)$$

From 5.84 $C_1 = C_3$, substituting C_1 into 5.85

$$\frac{1}{2}h^2C_3 + \frac{1}{2}(h)^2C_3 = 1,$$

$$C_3 = \frac{1}{h^2},$$

$$C_1 = \frac{2}{h^2},$$

substituting C_3 and C_1 into 5.83

$$C_2 = -\frac{1}{h^2} - \frac{2}{h^2} = -\frac{3}{h^2},$$

Results of difference approximation for u_{xx} is given below

$$v_{xx} = \frac{1}{h^2}v_{i-1,j} - \frac{2}{h^2}v_{i,j} + \frac{1}{h^2}v_{i+1,j},$$



Linear combination of v_{yy} with coefficient C_4, C_5, C_6 is given below

$$\begin{aligned} v_{yy} &= C_4 v(x, y + h_1) + C_5 v(x, y) + C_6 v(x, y - a_4 h_1), \\ v(x, y + h_1) &= v + h_1 v_y + \frac{1}{2} h_1^2 v_{yy} + \dots \\ v(x, y - a_5 h_1) &= v - a_5 h_1 v_y + \frac{1}{2} (a_5 h_1)^2 v_{yy} + \dots \end{aligned}$$

$$C_4(v + h_1 v_y + \frac{1}{2} h_1^2 v_{yy} + \dots) + C_5 v + C_6(v - a_5 h_1 v_y + \frac{1}{2} (a_5 h_1)^2 v_{yy} + \dots) = v_{yy},$$

We form linear combination with coefficient C_1, C_2, C_3 to obtain the following

$$(C_4 + C_5 + C_6)v + (h_1 C_5 - a_4 h_1 C_6)v_y + (\frac{1}{2} h_1^2 C_4 + \frac{1}{2} (a_5 h_1)^2 C_6)v_{yy} = v_{yy},$$

$$C_4 + C_5 + C_6 = 0, \quad (5.86)$$

$$h_1 C_4 - a_5 h_1 C_6 = 0, \quad (5.87)$$

$$\frac{1}{2} h_1^2 C_4 + \frac{1}{2} (a_5 h_1)^2 C_6 = 1 \quad (5.88)$$

From 5.87 $C_4 = a_5 C_6$, substituting C_4 into 5.88

$$\frac{1}{2} a_5 h_1^2 C_6 + \frac{1}{2} (a_5 h_1)^2 C_6 = 1,$$

$$C_4 = \frac{2}{h_1^2(1 + a_5)},$$

$$C_6 = \frac{2}{a_5 h_1^2(1 + a_4)},$$

substituting C_4 and C_6 into 5.86

$$C_5 = -\frac{1}{h_1^2(1 + a_5)} - \frac{1}{a_5 h_1^2(1 + a_5)} = -\frac{2}{a_5 h_1^2},$$

Results of difference approximation for u_{xx} is given below

$$v_{yy} = \frac{2}{h_1^2(1 + a_5)} v_{i,j+1} - \frac{2}{a_5 h_1^2} v_{i,j} + \frac{1}{a_5 h_1^2(1 + a_5)} v_{i,j-1}.$$

Velocity v at point 41

$$v_{i-1,j} - (a_5 + 1)\left(\frac{1}{a_5} + \lambda\right)v_{i,j} + \frac{1}{a_5} v_{i+1,j} - \frac{1}{2} \lambda (a_5 + 1) v_{i,j+1} + \frac{1}{2} \lambda (a_5 + 1) v_{i,j-1} = 0. \quad (5.89)$$



Velocity v at point 56

$$v_{i+1,j} - (a_5 + 1)\left(\frac{1}{a_5} + \lambda\right)v_{i,j} + \frac{1}{a_5}v_{i-1,j} - \frac{1}{2}\lambda(a_5 + 1)v_{i,j+1} + \frac{1}{2}\lambda(a_5 + 1)v_{i,j-1} = 0. \quad (5.90)$$

Velocity v at point 45 Linear combination of v_{xx} with coefficient C_1, C_2, C_3 is given below

$$\begin{aligned} v_{xx} &= C_1v(x-h, y) + C_2v(x, y) + C_3v(x+h, y), \\ v(x-h, y) &= v - hv_x + \frac{1}{2}h^2v_{xx} + \dots \\ v(x+h, y) &= v + hv_x + \frac{1}{2}(h)^2v_{xx} + \dots \end{aligned}$$

We form linear combination with coefficient C_1, C_2, C_3 to obtain the following

$$C_1(v - hv_x + \frac{1}{2}h^2v_{xx} + \dots) + C_2v + C_3(v + hv_x + \frac{1}{2}(h)^2v_{xx} + \dots) = v_{xx},$$

$$(C_1 + C_2 + C_3)v + (-hC_1 + hC_3)v_x + (\frac{1}{2}h^2C_1 + \frac{1}{2}(h)^2C_3)v_{xx} = v_{xx},$$

$$C_1 + C_2 + C_3 = 0, \quad (5.91)$$

$$-hC_1 + hC_3 = 0, \quad (5.92)$$

$$\frac{1}{2}h^2C_1 + \frac{1}{2}(h)^2C_3 = 1, \quad (5.93)$$

From 5.92 $C_1 = C_3$, substituting C_1 into 5.93

$$\frac{1}{2}h^2C_3 + \frac{1}{2}(h)^2C_3 = 1,$$

$$C_3 = \frac{1}{h^2},$$

$$C_1 = \frac{2}{h^2},$$

substituting C_3 and C_1 into 5.91

$$C_2 = -\frac{1}{h^2} - \frac{2}{h^2} = -\frac{3}{h^2},$$

$$v_{xx} = \frac{1}{h^2}v_{i-1,j} - \frac{3}{h^2}v_{i,j} + \frac{1}{h^2}v_{i+1,j},$$



Linear combination of v_{yy} with coefficient C_4, C_5, C_6 is given below

$$v_{yy} = C_4v(x, y + h_1) + C_5v(x, y) + C_6v(x, y - a_6h_1),$$

$$v(x, y + h_1) = v + h_1u_y + \frac{1}{2}h_1^2v_y + \dots$$

$$v(x, y - a_6h_1) = v - a_6h_1v_y + \frac{1}{2}(a_6h_1)^2u_{yy} + \dots$$

We form linear combination with coefficient C_1, C_2, C_3 to obtain the following

$$C_4(v + h_1v_y + \frac{1}{2}h_1^2v_y + \dots) + C_5v + C_6(v - a_6h_1v_y + \frac{1}{2}(a_6h_1)^2v_{yy} + \dots) = v_{yy},$$

$$(C_4 + C_5 + C_6)v + (h_1C_5 - a_6h_1C_6)v_y + (\frac{1}{2}h_1^2C_4 + \frac{1}{2}(a_6h_1)^2C_6)v_{yy} = u_{yy},$$

$$C_4 + C_5 + C_6 = 0, \tag{5.94}$$

$$h_1C_4 - a_6h_1C_6 = 0, \tag{5.95}$$

$$\frac{1}{2}h_1^2C_4 + \frac{1}{2}(a_6h_1)^2C_6 = 1, \tag{5.96}$$

From 5.95 $C_4 = a_6C_6$, substituting C_4 into 5.96

$$\frac{1}{2}a_6h_1^2C_6 + \frac{1}{2}(a_6h_1)^2C_6 = 1,$$

$$C_4 = \frac{2}{h_1^2(1 + a_6)},$$

$$C_6 = \frac{2}{a_6h_1^2(1 + a_6)},$$

substituting C_4 and C_6 into 5.94

$$C_5 = -\frac{1}{h_1^2(1 + a_6)} - \frac{1}{a_6h_1^2(1 + a_6)} = -\frac{2}{a_6h_1^2},$$

Results of difference approximation for u_{yy} is given below

$$v_{yy} = \frac{2}{h_1^2(1 + a_6)}v_{i,j+1} - \frac{2}{a_6h_1^2}v_{i,j} + \frac{1}{a_6h_1^2(1 + a_6)}v_{i,j-1}.$$

Velocity v at point 45

$$v_{i+1,j} - (a_6 + 1)(\frac{1}{a_6} + \lambda)v_{i,j} + \frac{1}{a_6}v_{i-1,j} - \frac{1}{2}\lambda(a_6 + 1)v_{i,j+1} + \frac{1}{2}\lambda(a_6 + 1)v_{i,j-1} = 0. \tag{5.97}$$



Velocity v at point 52

$$v_{i+1,j} - (a_6 + 1)\left(\frac{1}{a_6} + \lambda\right)v_{i,j} + \frac{1}{a_6}v_{i-1,j} - \frac{1}{2}\lambda(a_6 + 1)v_{i,j+1} + \frac{1}{2}\lambda(a_6 + 1)v_{i,j-1} = 0. \quad (5.98)$$

The discretisation of the velocity u using boundary conditions given below

$$\frac{\partial v}{\partial x}\Big|_{ver} = 0, v|_{\Omega_{hor}} = 0$$

thus

$$\frac{\partial v}{\partial x} = 0 \quad \text{for } x = 0 \quad \text{and} \quad x = N_x$$

$$v = 0 \quad \text{for } y = 0 \quad \text{and} \quad y = N_y$$

$$\frac{\partial v}{\partial x}\Big|_{0,j} = 0, \frac{\partial v}{\partial x}\Big|_{N_x,j} = 0. \quad v_{i,0} = 0 \quad \text{and} \quad v_{i,N_y} = 0$$

5.2.3 The pressure p at irregular points is calculated below.

Pressure p at points 29 and 65 Linear combination of p_x with coefficient C_1, C_2 is given below

$$p_x = c_1p(x - h, y) + c_2p(x + a_1h, y),$$

$$p(x - h, y) = p - hp_x$$

$$p(x + a_1h, y) = p + a_1hp_x,$$

We form linear combination with coefficient C_1, C_2 to obtain the following

$$c_1(p - hp_x) + c_2(p + a_1hp_x) = p_x,$$

$$(c_1 + c_2)p + (-hc_1 + a_1hc_2)p_x = p_x,$$

$$c_1 + c_2 = 0, \quad (5.99)$$

$$-hc_1 + a_1hc_2 = 1, \quad (5.100)$$

from 5.99 $c_1 = -c_2$

$$hc_2 + a_1hc_2 = 1,$$

$$(1 + a_1)hc_2 = 1,$$

$$c_2 = \frac{1}{h(1 + a_1)},$$



$$c_1 = \frac{1}{1 + a_1},$$

Results of difference approximation for p_x is given below

$$p_x = -\frac{1}{h(1 + a_1)}p_{i-1,j} + \frac{1}{h(1 + a_1)}p_{i+1,j}.$$

$$p_{i-1,j} - p_{i+1,j} = 0,$$

Pressure p at points 30 and 66 Linear combination of p_x with coefficient C_1, C_2 is given below

$$p_x = c_1p(x - h, y) + c_2p(x + a_2h, y),$$

$$p(x - h, y) = p - hp_x$$

$$p(x + a_2h, y) = p + a_2hp_x,$$

We form linear combination with coefficient C_1, C_2 to obtain the following

$$\begin{aligned} c_1(p - hp_x) + c_2(p + a_1hp_x) &= p_x, \\ (c_1 + c_2)p + (-hc_1 + a_2hc_2)p_x &= p_x, \end{aligned}$$

$$c_1 + c_2 = 0, \tag{5.101}$$

$$-hc_1 + a_2hc_2 = 1, \tag{5.102}$$

from 5.105 $c_1 = -c_2$

$$hc_2 + a_2hc_2 = 1,$$

$$(1 + a_2)hc_2 = 1,$$

$$c_2 = \frac{1}{h(1 + a_2)},$$

$$c_1 = \frac{1}{1 + a_2},$$

Results of difference approximation for p_x is given below

$$p_x = -\frac{1}{h(1 + a_2)}p_{i-1,j} + \frac{1}{h(1 + a_2)}p_{i+1,j}.$$

$$p_{i-1,j} - p_{i+1,j} = 0,$$

Pressure p at points 36 and 53 Linear combination of p_x with coefficient C_1, C_2 is given below

$$p_x = c_1p(x - h, y) + c_2p(x + a_3h, y),$$



$$p(x - h, y) = p - hp_x$$

$$p(x + a_3h, y) = p + a_3hp_x,$$

We form linear combination with coefficient C_1, C_2 to obtain the following

$$c_1(p - hp_x) + c_2(p + a_3hp_x) = p_x,$$

$$(c_1 + c_2)p + (-hc_1 + a_3hc_2)p_x = p_x,$$

$$c_1 + c_2 = 0, \tag{5.103}$$

$$\tag{5.104}$$

$$-hc_1 + a_3hc_2 = 1, \tag{5.105}$$

from 5.105 $c_1 = -c_2$

$$hc_2 + a_3hc_2 = 1,$$

$$(1 + a_3)hc_2 = 1,$$

$$c_2 = \frac{1}{h(1 + a_3)},$$

$$c_1 = \frac{1}{1 + a_3},$$

$$p_x = -\frac{1}{h(1 + a_3)}p_{i-1,j} + \frac{1}{h(1 + a_3)}p_{i+1,j},$$

$$p_{i-1,j} - p_{i+1,j} = 0,$$

Pressure p at points 41 and 56 Linear combination of p_x with coefficient C_1, C_2 is given below

$$p_x = c_1p(x - h, y) + c_2p(x + h, y),$$

$$p(x - h, y) = p - hp_x,$$

$$p(x + h, y) = p + hp_x,$$

$$c_1(p - hp_x) + c_2(p + hp_x) = p_x,$$

$$(c_1 + c_2)p + (-hc_1 + hc_2)p_x = p_x,$$

$$c_1 + c_2 = 0, \tag{5.106}$$

$$\tag{5.107}$$

$$-hc_1 + hc_2 = 1, \tag{5.108}$$



from 5.106 $c_1 = -c_2$

$$hc_2 + hc_2 = 1,$$

$$hc_2 = 1,$$

$$c_2 = \frac{1}{h},$$

$$c_1 = -\frac{1}{h},$$

Results of difference approximation for p_x is given below

$$p_x = -\frac{1}{h}p_{i-1,j} + \frac{1}{h(1+a_3)}p_{i+1,j}.$$

$$p_{i-1,j} - p_{i+1,j} = 0,$$

Pressure p at points 45 and 52 Linear combination of p_x with coefficient C_1, C_2 is given below

$$p_x = c_1p(x-h, y) + c_2p(x+h, y),$$

$$p(x-h, y) = p - hp_x,$$

$$p(x+h, y) = p + hp_x,$$

We form linear combination with coefficient C_1, C_2 to obtain the following

$$c_1(p - hp_x) + c_2(p + hp_x) = p_x,$$

$$(c_1 + c_2)p + (-hc_1 + hc_2)p_x = p_x,$$

$$c_1 + c_2 = 0, \tag{5.109}$$

$$-hc_1 + hc_2 = 1, \tag{5.110}$$

from 5.109 $c_1 = -c_2$

$$hc_2 + hc_2 = 1,$$

$$hc_2 = 1,$$

$$c_2 = \frac{1}{h},$$

$$c_1 = -\frac{1}{h},$$

Results of difference approximation for p_x is given below

$$p_x = -\frac{1}{h}p_{i-1,j} + \frac{1}{h(1+a_3)}p_{i+1,j}.$$

$$p_{i-1,j} - p_{i+1,j} = 0.$$

5.3 Results and Discussion

Below we present the results of the simulated Stokes flows calculated based on the iteration of the boundary conditions method and the finite difference method, for various flow parameters. This is done in a rectangular region and then extended into a region with a curved boundary at the bottom of the flow.

To illustrate the correctness of our numerical method, we compare certain key flow parameter obtained by the two methods. The identified key flow properties that we compare here are velocities, pressure distribution and the stream lines.

In the analysis of the flows, particular attention is given to the domain extension parameter, d . It must be noted also that in our case the viscous forces are stronger than the inertial forces and hence the velocities of the flow are expected to be small. Our main results are derived from the case of the flow region with a curved boundary at the bottom.

All the data that we are using here were obtained through MATLAB codes developed by the author. The codes are found under appendix in this project.

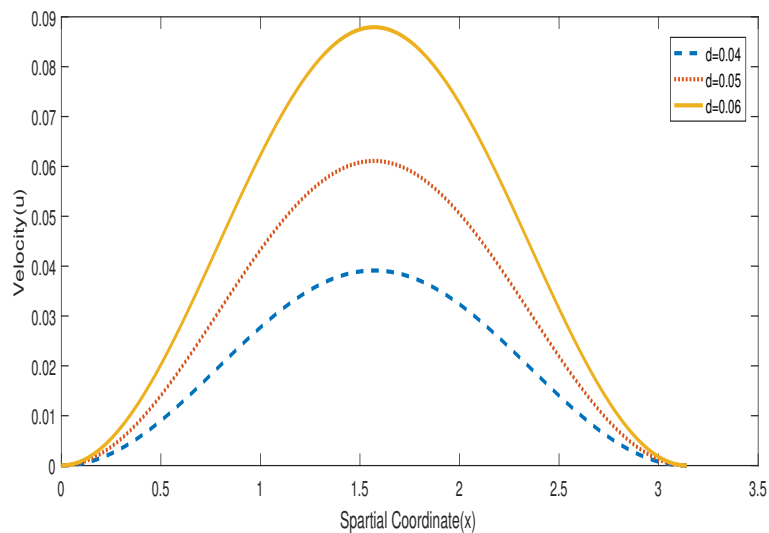


Figure 5.3: Horizontal velocity graph in the rectangular case

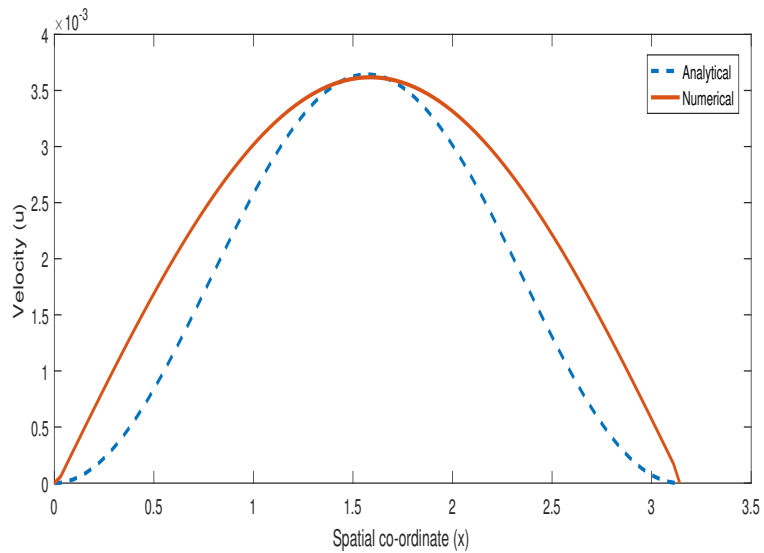


Figure 5.4: Horizontal velocity by the iteration of boundary conditions method (Analytical method) as compared to that by the finite difference method (numerical method) on rectangular domain when $d = 0.01219$.

Range	Maximum value of Absolute error
0 – 1.4	0.00087
1.4 – 1.7	0.00003
1.7 – 3.3	0.00097

Table 5.1: For range and maximum values of absolute error for horizontal velocity in rectangular domain

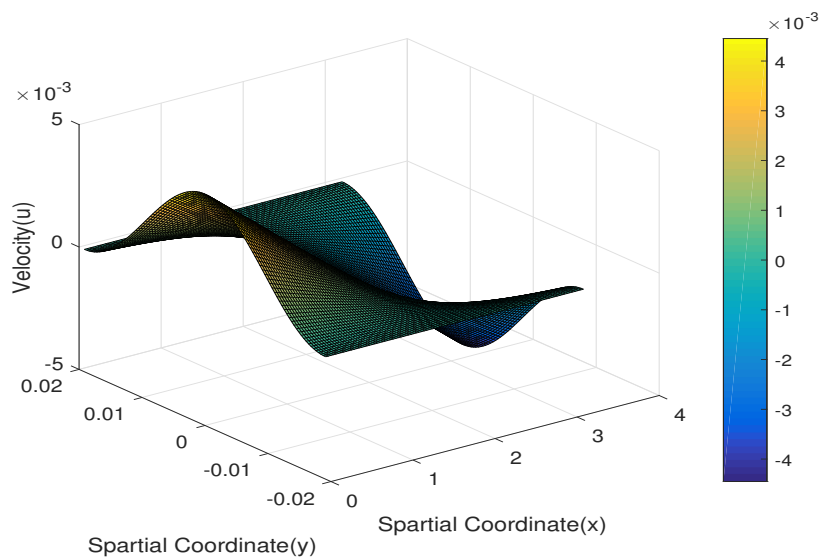


Figure 5.5: Horizontal velocity (u) in 2-dimensions over a rectangular region

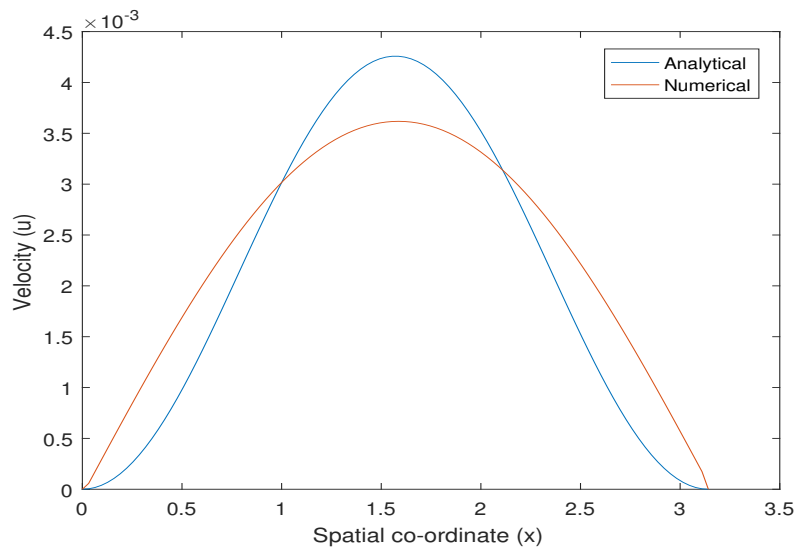


Figure 5.6: Horizontal velocity by the iteration of boundary conditions method (Analytical method) as compared to that by the finite difference method (numerical method) on rectangular domain when $d = 0.01319$.

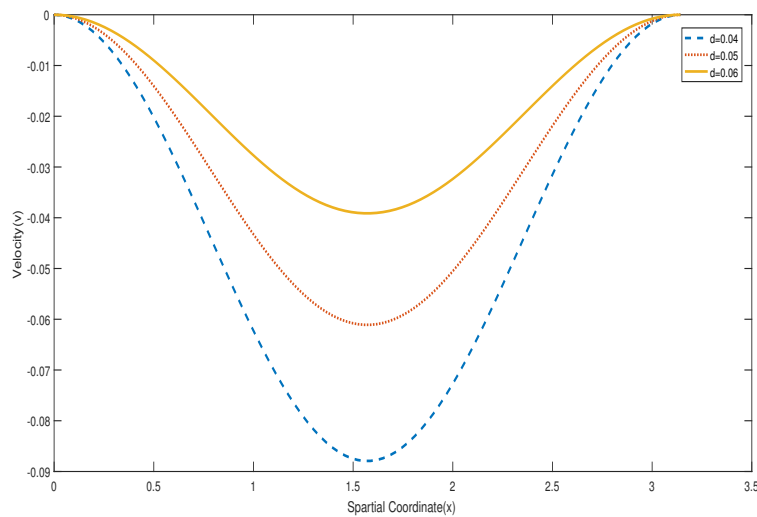


Figure 5.7: Shows the velocity profile (v) on rectangular domain

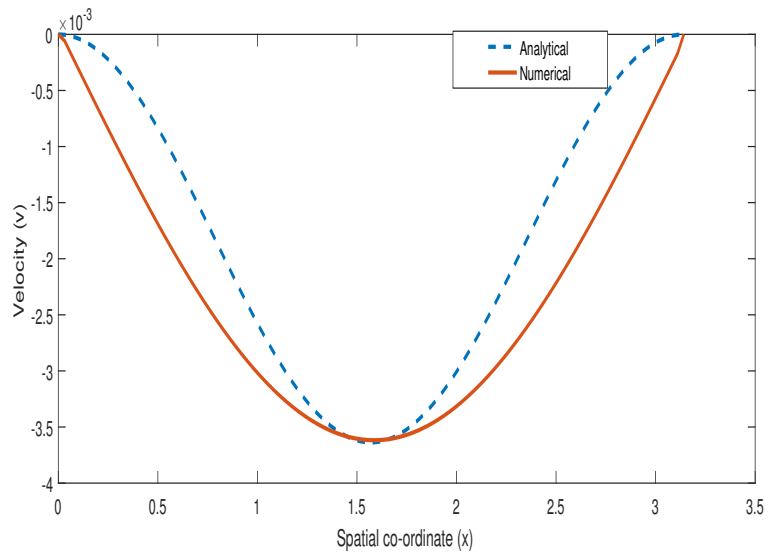


Figure 5.8: Shows the velocity profile(v) of Analytical and Numerical methods on rectangular domain when $d = 0.01219$

Range	Maximum value of Absolute error
0 – 1.4	0.00087
1.4 – 1.7	0.00003
1.7 – 3.3	0.00097

Table 5.2: For range and maximum value of absolute error for vertical velocity in rectangular domain

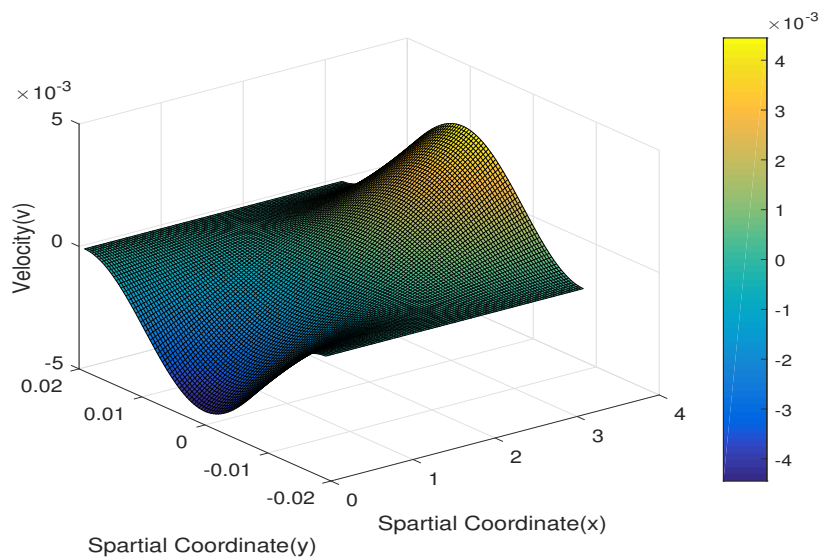


Figure 5.9: Velocity distribution in 2-dimensions over a rectangular domain

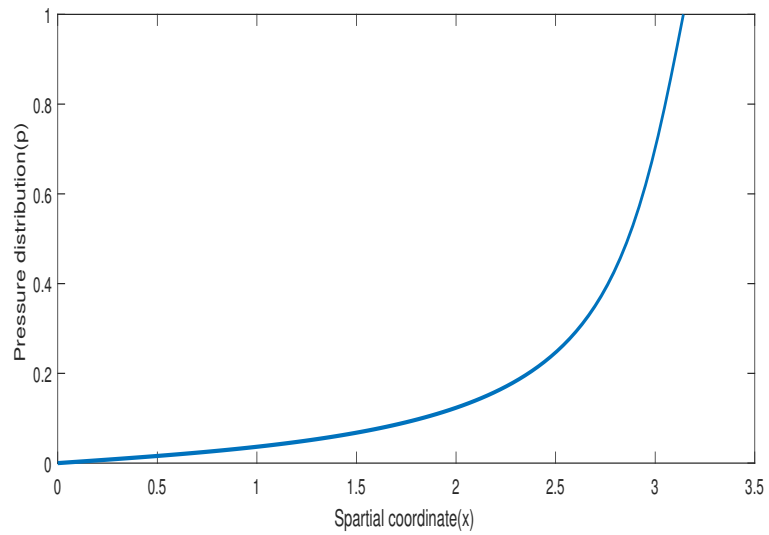


Figure 5.10: Pressure variation in the direction of the x-axis

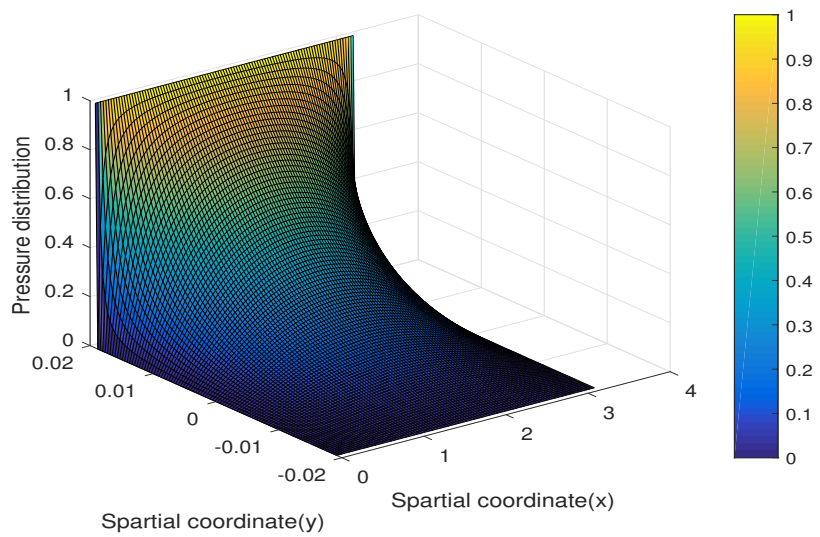


Figure 5.11: Pressure distribution in the flow over a rectangular domain

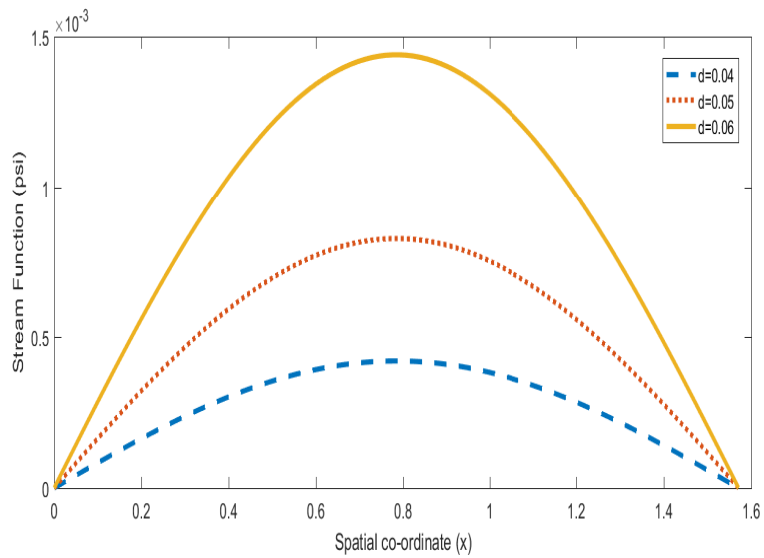


Figure 5.12: Stream function variation in the direction of the x-axis in the case of a rectangular domain

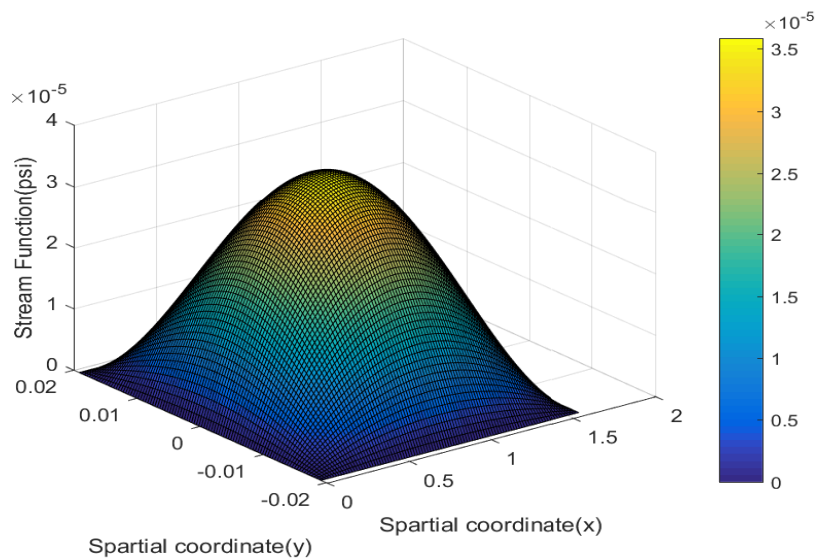


Figure 5.13: Graph of the stream function on a rectangular domain

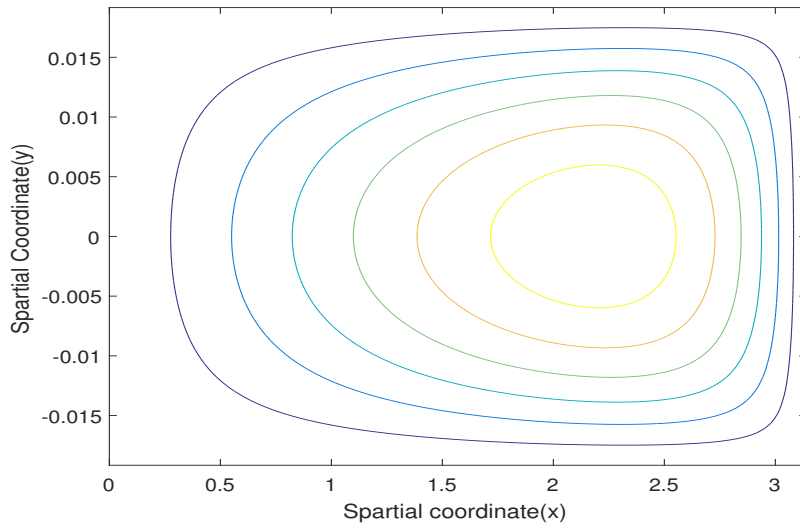


Figure 5.14: Contour lines for stream function by the Numerical method

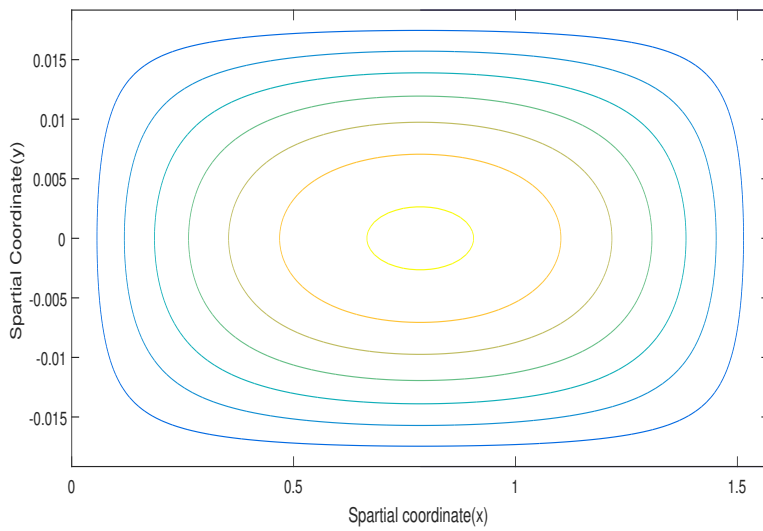


Figure 5.15: Contour lines for stream function by the Analytical method

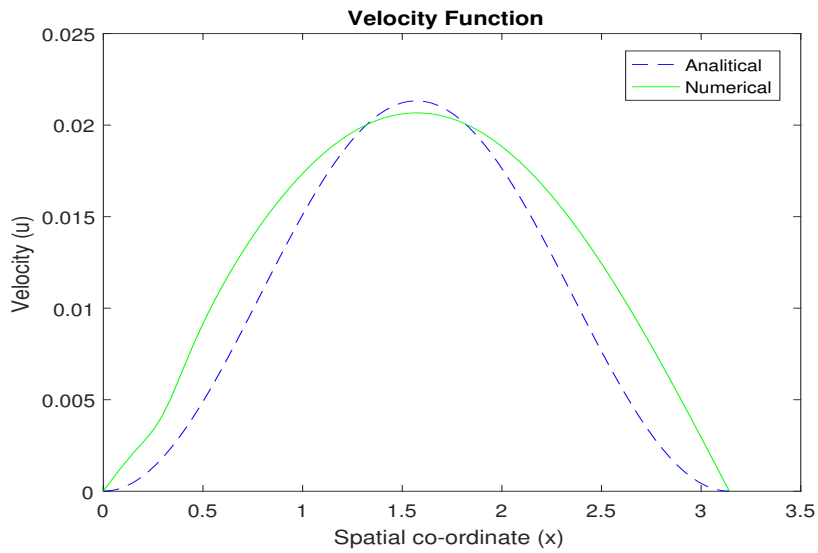


Figure 5.16: Horizontal velocity(u) for both analytical and numerical methods in the case of a curved boundary at the bottom of irregular domain when $d = 0.0779$

Range	Maximum value of Absolute error
0 – 1.4	0.00045
1.4 – 1.7	0.0005
1.7 – 3.3	0.0005

Table 5.3: For range and maximum values of absolute error for horizontal velocity in irregular domain

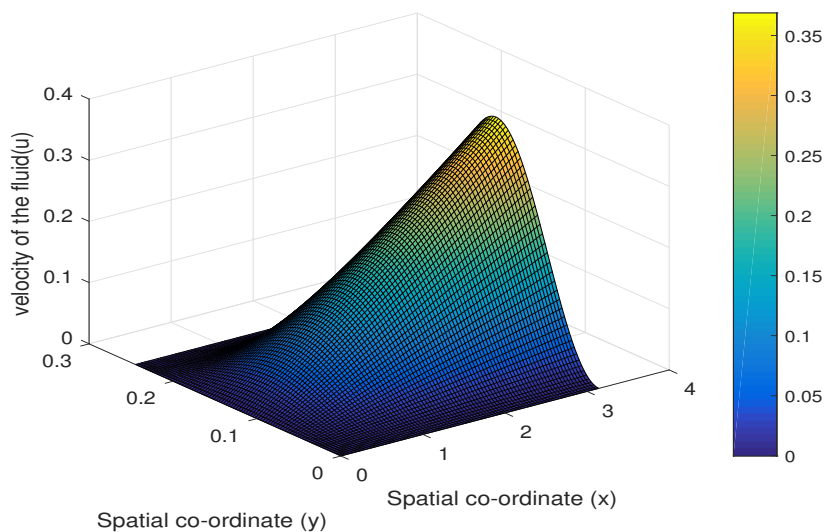


Figure 5.17: Shows the surface of velocity profile(u) on irregular domain

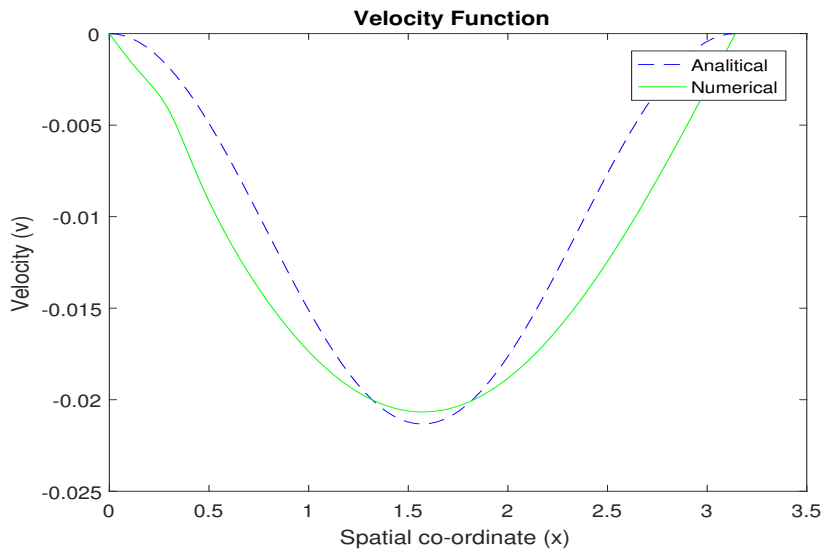


Figure 5.18: vertical velocity(u) for both analytical and numerical methods in the case of a curved boundary at the bottom of irregular domain when $d = 0.0779$

Range	Maximum value of Absolute error
0 – 1.4	0.00045
1.4 – 1.7	0.0005
1.7 – 3.3	0.0005

Table 5.4: For range and maximum values of absolute error for vertical velocity in irregular domain

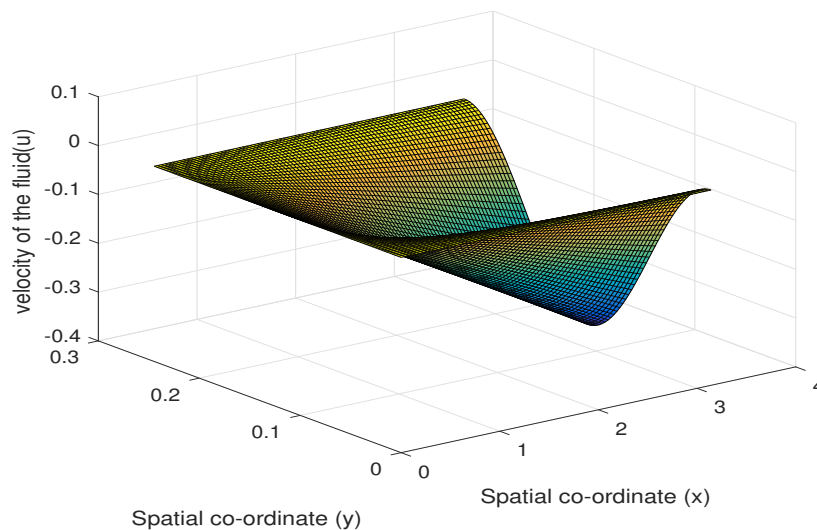


Figure 5.19: Shows the surface of velocity profile(v) on irregular domain

5.4 Discussion of the results

The dimensionless velocity profiles, stream lines and pressure distribution graphs for the Stokes flow problem are obtained, and their behavior for various flow parameters are discussed. The iteration of boundary conditions and the finite difference methods have been used to solve the Stokes flow problem in the rectangular and irregular boundary conditions cases. The domain extension parameter, d , has been tested for various values in order to have a clear insight into the behavior of the flow velocities, stream function behaviour and pressure distribution. The absolute errors obtained when comparing the analytical and numerical methods were presented in tabular form.

The runs for $d = 0.04, 0.05$ and 0.06 is done in the case of a rectangular domain by the method of Iteration of boundary conditions. The results are shown in the figure 5.3 and 5.7. We observed that the flow behaviour is dependent on the parameter d . In particular the vertical and horizontal velocities increase with the slight increase in parameter d . We also observed that the fluid flow is stationary at the boundaries of the domain and increases steadily as it approaches the centre of the flow. In figure 5.4 and figure 5.8, we observe that the two methods are qualitatively in good agreement. However, away from the centre of the flow numerical results over estimate the exact results. The difference between the analytical and numerical results is shown in Table 5.1 and 5.2.

Figure 5.5 and 5.9 illustrate how the velocities are distributed throughout the domain. Figure 5.6 shows how the velocities for numerical and analytical methods are changing when the value of d is changed by a small margin. The pressure in the fluid is increasing exponentially as the flow progresses as shown in Figure 5.10. The pressure distribution is illustrated by surface plot in Figure 5.11. The contour lines produced by the two methods are qualitatively in good agreement although something might have to be done for the minor discrepancies that are observed. The cases for $d = 0.04, 0.05$ and 0.06 are illustrated. It was also noted that the velocities produced by the analytical method are in good agreement with those of the finite difference method again we notice that away from the centre of the flow the numerical results over estimate the exact results, something that is observed in all our results. The difference between the two methods is shown in Table 5.3 and 5.4. The general observation is that, the numerical method has been shown to be a good approximation to the near exact method. Some discrepancies are observed requiring us to go over our numerical calculations with a view of reducing the errors between flow properties that seem to arise away from the center of the flow.

5.5 Conclusion

The Stokes flow problem in rectangular and irregular boundary conditions has been investigated through numerical and analytical approaches with a view of validating the numerical results produced by the finite difference method. It is concluded that the behavior of the fluid flow properties, such as velocity profile, stream function and pressure distribution depend solely on the domain extension (parameter d), where in quantities increase as the domain extension increases. The study confirms this feature in the case of the application of the numerical method. The results of finite difference numerical method are compared with those obtained by the analytical method to ascertain the correctness of the implementation of the

numerical scheme. The two methods were found to be qualitatively in good agreement. There are some discrepancies which may need to be accounted for. A revision of the implementation of the numerical scheme might provide the answers for the errors. The good thing about this work is that we have successfully implemented a numerical method which can then be used in place of the analytical method in practical situations. It is very well known that, it is a very difficult task to implement analytical solutions compared to numerical solutions. Thus for practical purposes, we use the numerical approach. It was important, therefore, for us to come up with a numerical algorithm, using the finite difference approach, which we tested on a rectangular domain as well as on a domain with a curved boundary. The implementation of all these approaches were done through MATLAB codes which were developed by the author. In future the author is planning to come up with a standard scheme and code for implementation of the numerical method for a general irregular domain.

Coding for Stokes flow problem in rectangular and Irregular domain.

Rectangular

```
%%%%%%%%%DISPLAY THE VELOCITY PROFILE, PRESSURE DISTRIBUTION AND STREAM LINES
%%%%%%%%%FUNCTION FOR VELOCITY ON RECTANGULAR DOMAIN%%%%%%%%%
```

```
%%% ANALYTICAL METHOD FOR SOLVING STOKES PROBLEM IN RECTANGULAR
%%%DOMAIN%%%
```

```
clc, clear all
%d=0.04; % DOMAIN EXTENSION
%d=0.05;
%d=0.06;
%d=0.2;
%d=0.01319;
%d=0.5;
%d=1;
%d=1.5
%d=2
d=0.01219;

nx=100; % No of nodes for x
ny=100; % No od nodes for y
a=0; % Starting values on x
b1=pi; % End value on x
x=linspace(a,b1,nx); % Interval on x
y=linspace(-d*(pi/2),d*(pi/2),ny); % Interval on y
a=zeros(nx,1); % Intialising function a
b=zeros(nx,1); % Intialising function b
v1=zeros(nx,ny); % Initialysng function b
m=1;
maxit=1;
while m<=maxit
for i=1:nx
for j=1:nx
```

%%%FUNCTION WITH DEFINES VELOCITY OF THE FLUID WHEN BOTH X AND Y ARE CHANGING%%%%%%%%%

```
a(i)=0;
```

```
b(i)=sqrt(2/pi)*(-1/m)*cos(m*(x(i)+pi/2));
```

```
% FUNCTION WHICH DESCRIBE THE FLUID FLOW
```

```
v1(i,j) = sqrt(2/pi).*(-1/m).*((b(i)-a(i))/2.*(sinh(m.*d*pi)+m.*d.*pi).*...
((2*sinh(m*pi*d/2)-m*pi*d.*cosh(m*pi*d/2)).*cosh(m.*y(j))+2*...
sinh(m*pi*d/2)*(m*y(j)*sinh(m*y(j))))+(b(i)-a(i))/2.*(sinh(m.*d.*pi)+...
m.*d.*pi).*(2*cosh(2*cosh(m*pi*d/2)-m*pi*d.*sinh(m*pi*d/2)).*sinh(m.*...
y(j))+2*cosh(m*pi*d/2).*(m.*y(j)).*cosh(m.*y(j)))).*sin(m.*(x(i)+pi));
```

```
end;
```

```
end;
```

```
m=m+1;
```

```
end;
```

```
%%% NUMERICAL METHOD FOR SOLVING STOKE'S PROBLEM IN RECTANGULAR
```

```
%%%DOMAIN%%%
```

```
%%
```

```
% Specifying parameters
```

```
nx=100;
```

```
% Number of steps in space(x
```

```
ny=100;
```

```
% Number of steps in space(y
```

```
% Number of iterations
```

```
a=0;
```

```
b1=pi;
```

```
h=(b1-a)/nx;
```

```
x=linspace(a,b1,nx);
```

```
y=linspace(-d*(pi/2),d*(pi/2),ny);
```

```
u=zeros(nx,ny);
```

```
% Preallocating u
```

```
v=zeros(nx,ny);
```

```
% Preallocating v
```

```
p=zeros(nx,ny);
```

```
% Preallocating p
```

```
w=zeros(nx,ny);
```

```
% Preallocating w
```

```
psi=zeros(nx,ny);
```

```
% Preallocating w
```

```
m=1;
```

```
niter=10000;
```

```
j=2:nx-1;
```

```
i=2:ny-1;
```

```
for it=1:niter
```

```
% Boundary Conditions
```

```
w(:,1)=0;
```

```

w(:,ny)=0;
w(1,:)=0;
w(nx,:)=1 ;

w(i,j)=(((w(i+1,j)+w(i-1,j)))+(w(i,j+1)+w(i,j-1))))/4;           % Vorticity function

end

%%
i=2:nx-1;
j=2:ny-1;
%%%%%%Explicit iterative scheme with C.D in space (5-point difference)
for it=1:niter

% Boundary Condition
psi(:,1)=0;
psi(:,ny)=0;
psi(1,:)=0;
psi(nx,:)=0;

psi(i,j)=(((psi(i+1,j)+psi(i-1,j)))+(psi(i,j+1)+psi(i,j-1)))+(h.*w(i,j)))/4; % Stream

end

%%
for i=2:nx-1
for j=2:ny-1

u(:,1)=0;
u(:,ny)=0;
u(1,:)=1;
u(nx,:)=1;
u(i,j)=(psi(i,j)-psi(i,j-1)+psi(i-1,j)-psi(i-1,j-1))./2.*h; % Horizontal Velocity of t
end
end
%%
for i=2:nx-1
for j=2:ny-1

v(:,1)=0;
v(:,ny)=0;
v(1,:)=0;
v(nx,:)=0;
v(i,j)=-((psi(i,j)-psi(i-1,j)+psi(i,j-1)-psi(i-1,j-1))./2.*h; % Vertical velocity of th
end
end

```

```

% Boundary Condition
p(:,1)=0;
p(:,ny)=1;
p(1,:)=0;
p(nx,:)=0;

i=2:nx-1;
j=2:ny-1;
for it=1:niter

p(i,j)=(1/4)*(p(i+1,j)+p(i-1,j)+p(i,j+1)+p(i,j-1));           % Pressure function of the
end

set(gcf, 'Renderer', 'zbuffer');

figure(1)
plot(x,-v1(:,10))
%title('Velocity Function ')
xlabel('Spatial co-ordinate (x)')
ylabel('Velocity (u)')
%legend('Velocity function')
zlabel('velocity of the fluid(u)')
hold all

figure(1)
plot(x,-v(10,:))
%title('Velocity Function ')
xlabel('Spatial co-ordinate (x)')
ylabel('Velocity (u)')
%legend('Velocity function')
zlabel('velocity of the fluid(v)')
hold all
g=abs(v1(:,10)-(v(10,:)))';
figure(2)
plot(y,g)
xlabel('Spatial co-ordinate (x)')
ylabel('Absolute error')

figure(3)
plot(x,p(10,:))
title('Pressure distribution ')
xlabel('Spatial co-ordinate (x)')
ylabel('Pressure (p)')

figure(4)
surf(x,y,p)
title('Pressure distribution ')

```



```

psi=zeros(nx,ny); % Preallocating w
niter=10000;
m=1;
while m<=maxit
for i=1:nx
for j=1:nx

%%%%%% FUNCTION WHICH DEFINES VELOCITY OF THE FLUID WHEN BOTH X AND Y ARE CHANGING%%%%%%%%

a(i)=0; % Function which describe the fluid flow
b(i)=sqrt(2/pi)*(-1/m)*cos(m*(x(i)+pi/2)); % Function which describe the fluid flow

% Function which describe the fluid flow
u2(i,j) = sqrt(2/pi).*(1/m).*((b(i)-a(i))/2.*(sinh(m.*d*pi)+m.*d.*pi).*...
((2*sinh(m*pi*d/2)-m*pi*d.*cosh(m*pi*d/2)).*cosh(m.*y(j))+...
2*sinh(m*pi*d/2)*(m*y(j)*sinh(m*y(j))))+(b(i)-a(i))/2.*...
(sinh(m.*d.*pi)+m.*d.*pi).*(2*cosh(2*cosh(m*pi*d/2)-m*pi*d.*...
sinh(m*pi*d/2)).*sinh(m.*y(j))+2*cosh(m*pi*d/2).*(m.*y(j)).*...
cosh(m.*y(j)))).*sin(m.*(x(i)+pi));
end;
end;
m=m+1;
end;

% Define variables

Nx=85; % No of nodes in x direction
Ny=85; % No of nodes in y direction
tol=1d-6; % Tolerance
err=1; % Error
k=0; % Iteration counter
%x=linspace(0,pi,Nx);
r=pi/2;
h1=0;
% y1=linspace(pi/4,pi/2,27);
% x1=linspace(0,pi/4,29);
% x2=k+sqrt(r.^2-(y1-h1).^2);
% x3=linspace(pi/2,pi,29);
% x=[x1 x2 x3];
y=linspace(0,d*pi,Ny);

% Initialise the Matrix

a1=4-sqrt(15);
a2=4-2*sqrt(3);
a3=3-sqrt(7);

```

```

a4=3-sqrt(7);
a5=a2;
a6=a1;
h1=pi/16;
h2=d*pi/8;
lapda=(h1/h2);
u=zeros(Nx,Ny);
u(:,Ny)=1;
ukp1=u;
%Output Column
% fprintf('K |')
% for i=1:Nx-2
%     for i=1:Ny-2
%         fprintf('u(%li,%li)|', i,j)
%     end
% end
% fprintf('error\n')
%%%ITERATION USING JACOBBI METHOD UNTIL UNTIL IT CONVERGENCE%%%%%%%%%
while err>tol
%%%%%%%%%%%%%UPDATE ITERATION NUMBER%%%%%%%%%%%%%

k=k+1;
%fprintf('%4i |',k)
%Loop through computational nodes
for i=2:Nx-1
for j=2:Ny-1
ukp1(5,1)=0;    ukp1(6,1)=0; ukp1(7,1)=0; ukp1(8,1)=0; ukp1(9,1)=0;
ukp1(10,1)=0; ukp1(11,1)=0;
ukp1(5,2)=0;    ukp1(6,2)=0; ukp1(7,2)=0; ukp1(8,2)=0; ukp1(9,2)=0;
ukp1(10,2)=0; ukp1(11,2)=0;
ukp1(6,3)=0; ukp1(7,3)=0; ukp1(8,3)=0; ukp1(9,3)=0;
ukp1(10,3)=0;
ukp1(8,4)=0;

ukp1(i,j)=0.25*(u(i,j+1)+u(i+1,j)+u(i,j-1)+u(i-1,j));
% fprintf('%8,6F|',ukp1(i,j))
%At point 29
ukp1(4,1)=1/((1+a1)*(1/a1+lapda))*(u(3,1)+1/2*lapda*(1+a1)*u(4,2));
%     %At point 30
ukp1(4,2)=1/((1+a2)*(1/a2+lapda))*(u(3,2)+1/2*lapda*(1+a2)*u(4,3)+1/2*
lapda*(1+a2)*u(4,1));
%     %At point 36
ukp1(5,3)=1/((1+a2)*(1/a2+lapda))*(u(5,7)+((1+a4)/(1+a3))*u(5,2));
%At point 41
ukp1(6,4)=1/(2*(1+lapda/a5))*(u(7,4)+u(5,4)+2*lapda/(1+a5)*u(6,5));
% %     %At point 45
ukp1(7,4)=1/(1+lapda/a6)*(u(6,4)+2*lapda/(1+a6)*u(7,5));

```

```

%      %At point 52
ukp1(9,4)=1/(2*(1+lapda/a6))*(u(10,4)+2*lapda/(1+a6)*u(9,5));
%At point 56
ukp1(10,4)=1/(2*(1+lapda/a5))*(u(11,4)+u(9,4)+2*lapda/(1+a5)*u(10,5));
%At point 60
ukp1(11,3)=1/((1+a2)*(1/a2+lapda))*(u(12,7)+((1+a4)/(1+a3))*u(11,2));
%At point 65
ukp1(12,1)=1/((1+a2)*(1/a1+lapda))*(u(13,1)+1/2*lapda*(1+a1)*u(12,2));
%      %At point 66
ukp1(12,2)=1/((1+a2)*(1/a2+lapda))*(u(13,2)+1/2*lapda*(1+a2)*u(12,3)+
1/2*lapda*(1+a2)*u(12,1));
end
end
b=[0;0;0;0;0;0;lapda;0;0;0;0;0;0;lapda;0;0;0;0;0;0;lapda;0;0;0;0;0;0;
lapda;0;0;0;0;lapda;0;0;0;lapda;0;0;0;lapda;0;0;lapda;0;0;0;lapda;
0;0;0;lapda;0;0;0;0;lapda;0;0;0;0;0;0;lapda;0;0;0;0;0;0;lapda;0;0;
0;0;0;0;lapda;0;0;0;0;0;0;lapda];

%%%%%%%%%%%%%%CALCULATE ERROR%%%%%%%%%%%%%%

err=sqrt(sum(sum((ukp1-u).^2)));
%fprintf('%8.6f\n',err)
%update u
u=ukp1;
end
plot(x,u(:,6))

% set(gcf, 'Renderer', 'zbuffer');
% figure(1)
% plot(y,u(1,:))
% title('Velocity Function ')
% xlabel('Spatial co-ordinate (y)')
% ylabel('Velocity (u)')
% legend('Velocity function')
% zlabel('velocity of the fluid(u)')
% hold on
figure(1)
plot(x,u2(:,6),'b--')
hold all
figure(1)
plot(x,-u(:,6),'g')
xlabel('Spatial co-ordinate (y)')
ylabel('Velocity (v)')
legend('Analitical','Numerical')
zlabel('velocity of the fluid(u)')
hold all
figure(2)
plot(x,abs(u(:,6)+u2(:,6)))

```

```
xlabel('Spatial co-ordinate (x)')  
ylabel('Absolute error')  
  
figure(3)  
surf(x,y,u2)  
xlabel('Spatial co-ordinate (x)')  
ylabel('Spatial co-ordinate (y)')  
zlabel('velocity of the fluid(u)')
```

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